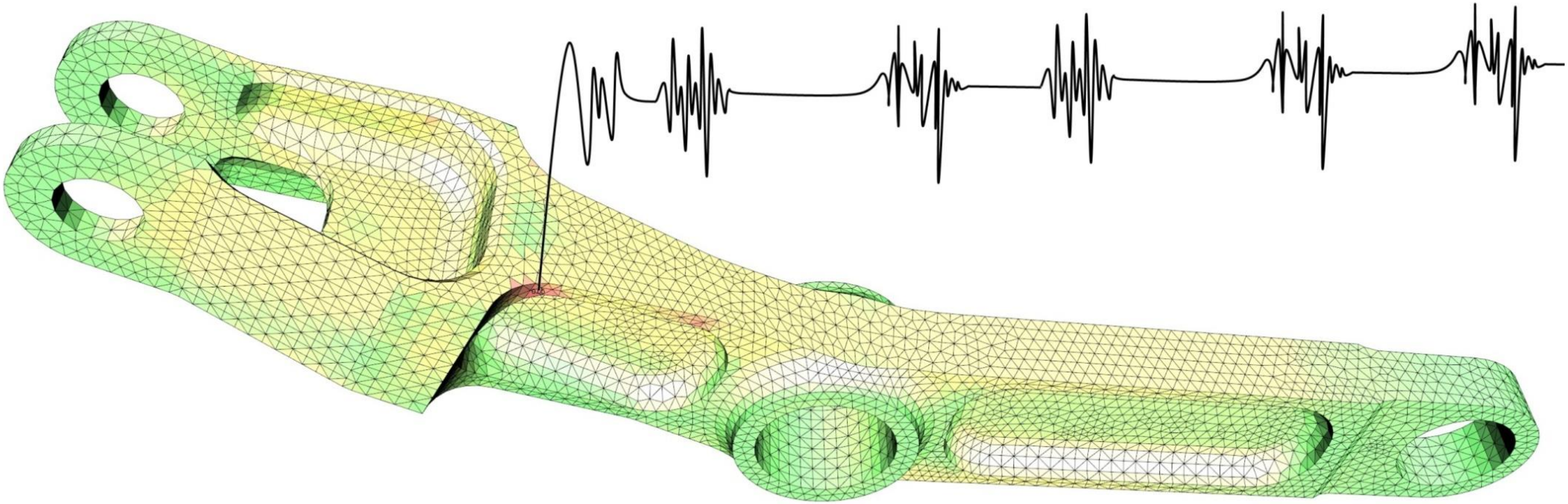


MULTIAXIAL FATIGUE

An insight from the application standpoint

Emphasis to some FEM assisted fatigue analysis aspects

DOMENICO QUARANTA
CHIEF STRESS ENGINEER (TRAINERS DPT) AND FDT SPECIALISTS COORDINATOR – PILATUS AIRCRAFT LTD

ABOUT THE LECTURER: DOMENICO QUARANTA

- 1998, MSc Aerospace Engineering at Politecnico di Torino
- *Stress Engineer* - Partenership Alenia-NASA: ISS Node 2 MDPS
- *Stress Engineer* - Aermacchi M346 FWD Fuselage
- *Senior Stress Engineer* - Airbus A380 Trent 900 Nacelles
- *Senior Stress Engineer* - Airbus A380 Main Deck Cargo Door
- *Chief Stress Engineer Trainers/Coordinator Fatigue Specialists* - Pilatus Aircraft (PC-9, PC-9(M), PC-7MkII, PC-21)
- *Lecturer - Fatigue-Damage Tolerance* at Universities of Zürich, Brescia, Parma, Milano, Napoli

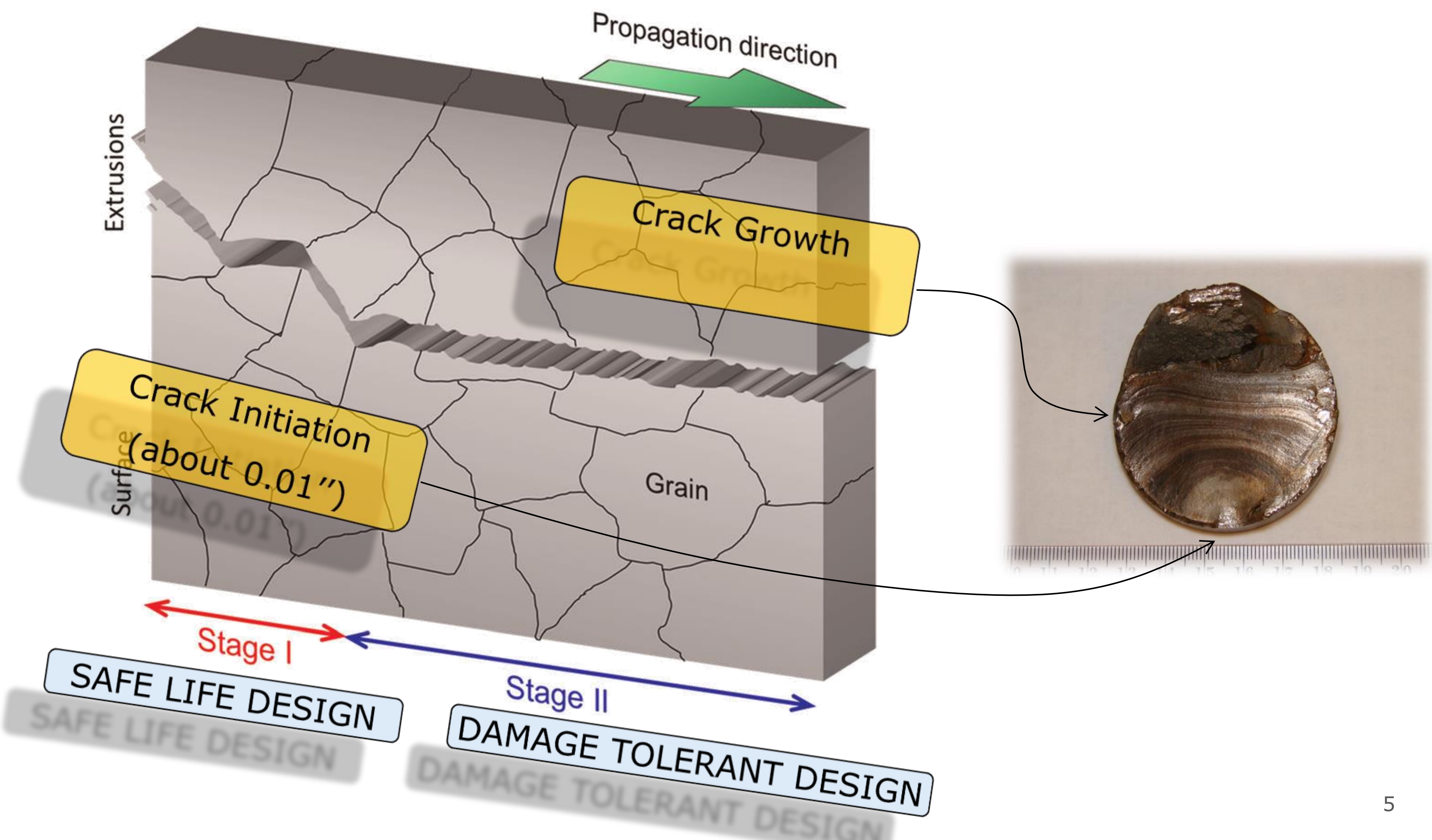
OUTLOOK

- Introduction - Uniaxial Fatigue Analysis Process
  - Fatigue Models: Uniaxial Stress Based and Strain Based Fatigue
  - Damage Cumulation Rule, Cycles Definition and Counting, Stress Concentration and Notch Factors
  - Neuber and Glinka elastic-plastic stress calculation
- Multiaxial (*FEM assisted*) Fatigue Analysis
  - Stress Tensors and Time History Assembling
  - Multiaxial Filtering (Racetrack)
  - Proportional and Non-Proportional Loadings
  - Elastic-Plastic Stress Tensor Time History calculation in case of Proportional Loading
  - Elastic-Plastic Stress Tensor Time History calculation in case of Non Proportional Loading
  - Cyclic Plasticity
  - Simplified approach: 'Proportional Reduction'
  - LCF Critical Plane Approaches with Smith-Watson-Topper, Brown-Miller, Fatemi-Socie
  - HCF Dang-Van Approach

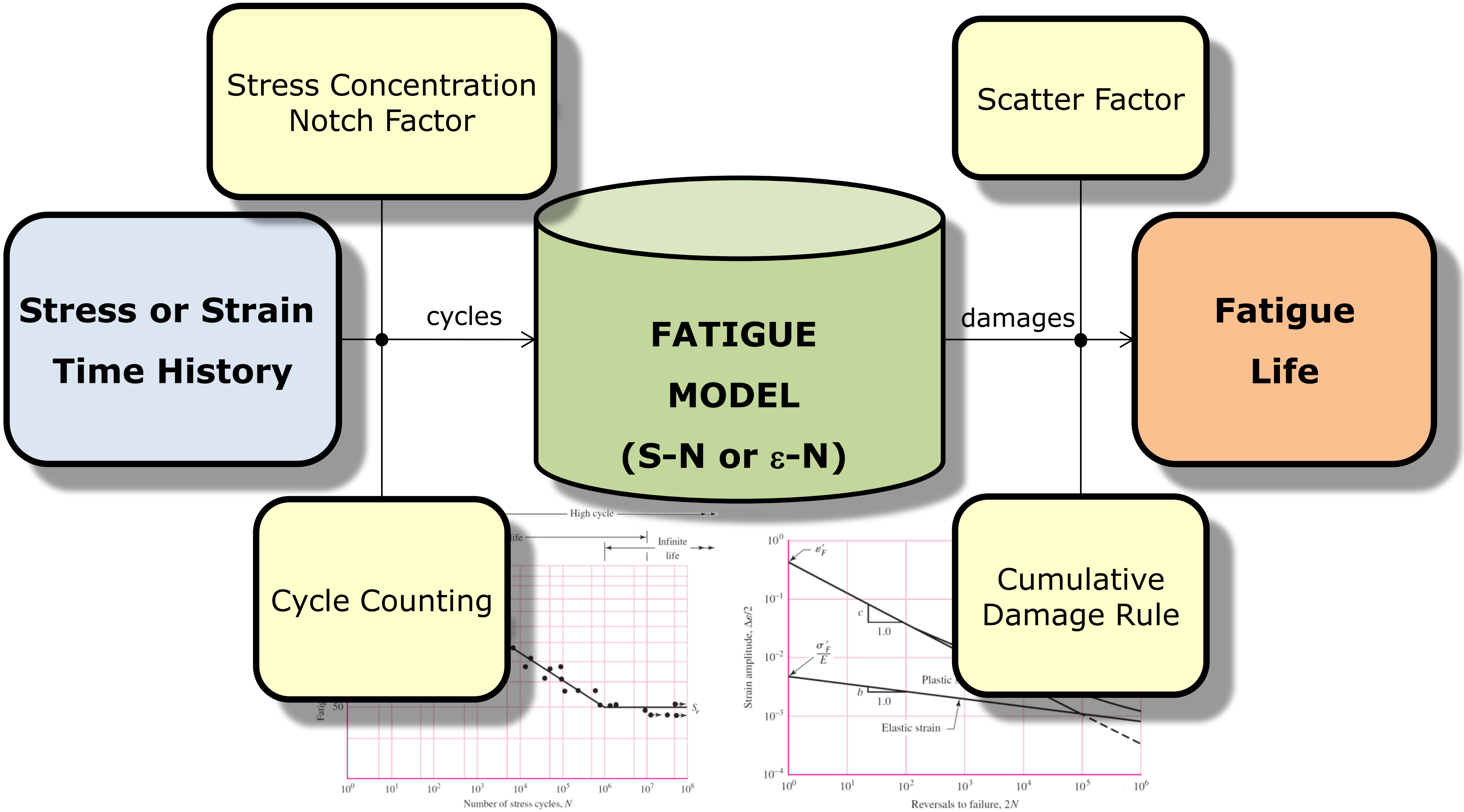
FATIGUE IN METALLIC COMPONENTS

- Fatigue is the progressive and localized structural damage that occurs when a material is subjected to cyclic loading.
- If the cyclic stresses are above a certain threshold value (endurance limit) (for most materials used in lightweight structures the threshold is 0), microscopic cracks nucleate (generally at notches, where there are stress concentrations) after a certain number of cycles.
- Once nucleated, the crack grows up to the critical size, at which the structure suddenly collapses (the remaining section cannot withstand statically the applied cyclic load).
- The fatigue failure occurs at cyclic stress levels which are below the allowable static stress.

FATIGUE IN METALLIC COMPONENTS



FATIGUE IN METALLIC COMPONENTS



FATIGUE MODELS

- There are many fatigue methods that can be used. They belong to two different classes:

  - **Stress based S-N**

    (High Cycle Fatigue - HCF)

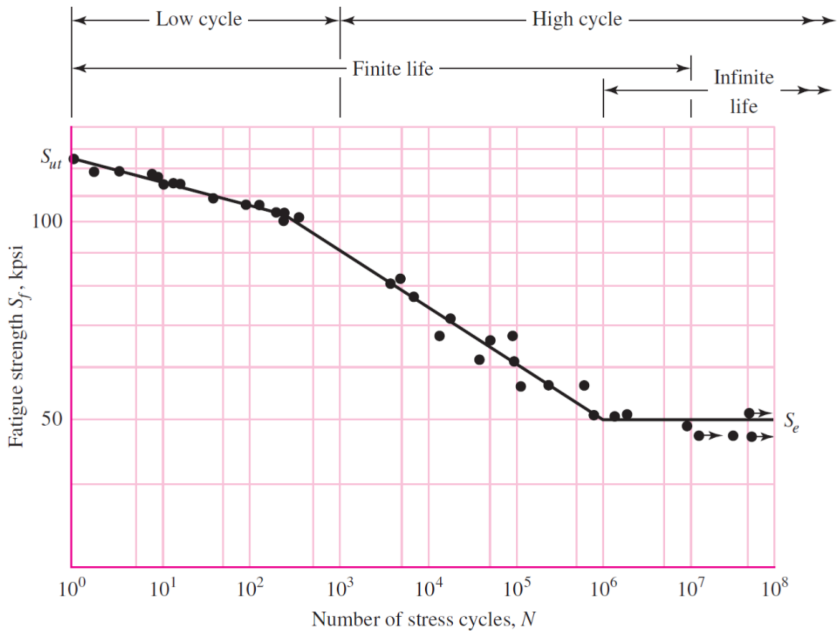
  - **Strain based ε-N**

    (Low Cycle Fatigue - LCF)

    (This is computationally more involving because elastic-plastic stress and strains have to be calculated)

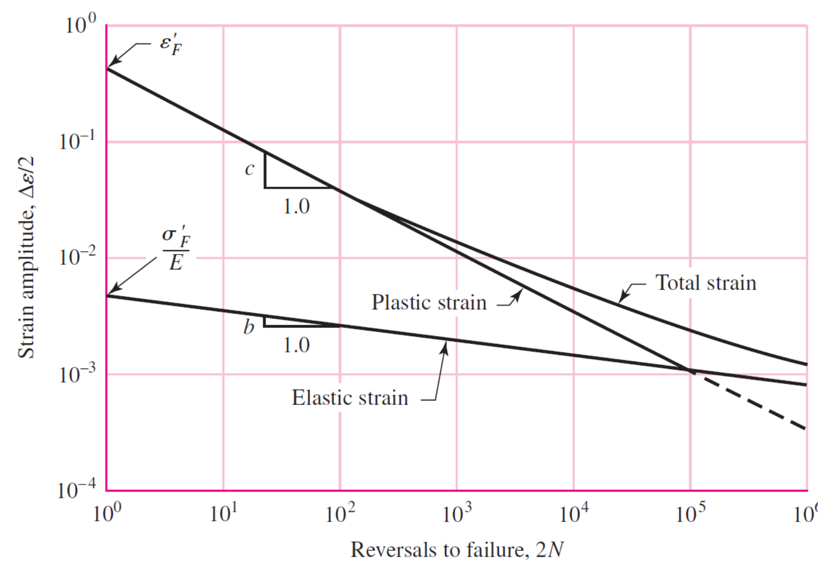
FATIGUE MODELS

- Stress based S-N curves and strain based ε-N curves



$$\frac{\Delta S}{2} = C \cdot N^m$$

Basquin Equation



$$\frac{\Delta\varepsilon}{2} = \frac{\sigma_f{}'}{E}(2N_f)^b + \varepsilon_f{}'(2N_f)^c$$

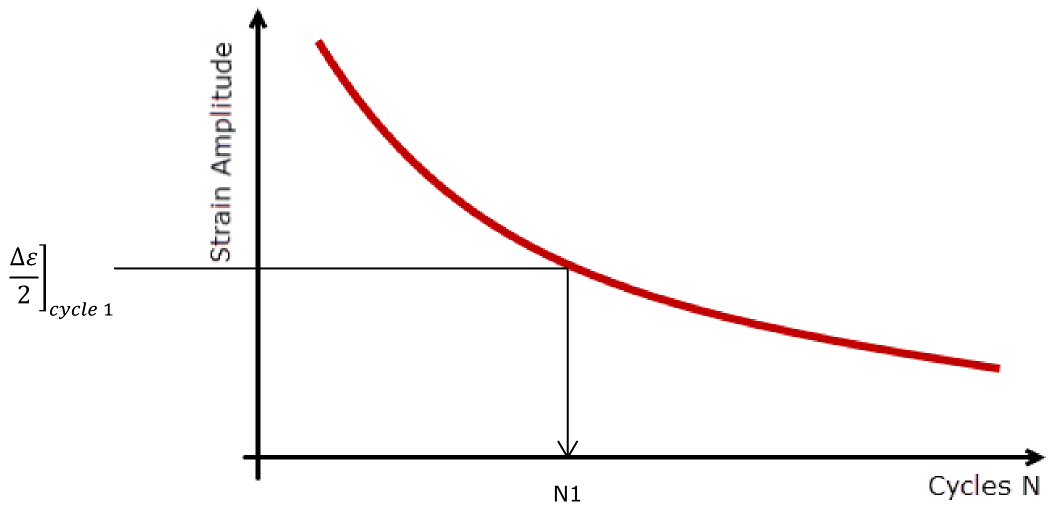
Coffin-Manson Equation

FATIGUE MODELS

- In both cases the curves and/or input parameter (stress amplitude or strain amplitude) must be modified in order to take into account
  - Mean Stress Effects
    - Goodman, Gerber, Soderberg, Walker, ... in case of HCF
    - Smith-Watson-Topper, Morrow, Manson-Halford, ... in case of LCF
  - Temperature effects
  - Surface conditions
  - Loading modes (in case of S-N curves)
  - Size effects (in case of S-N curves)
  - Reliability factor
  - ...

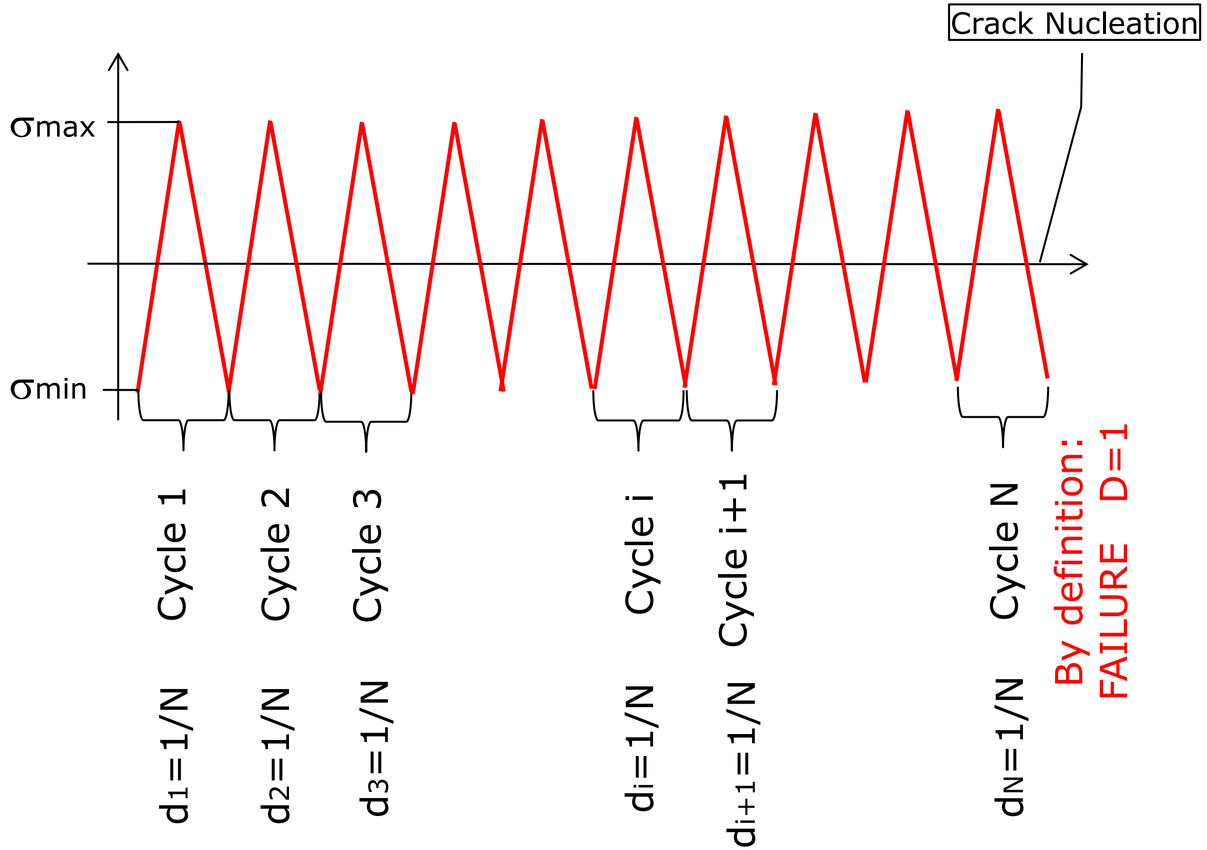
FATIGUE MODELS

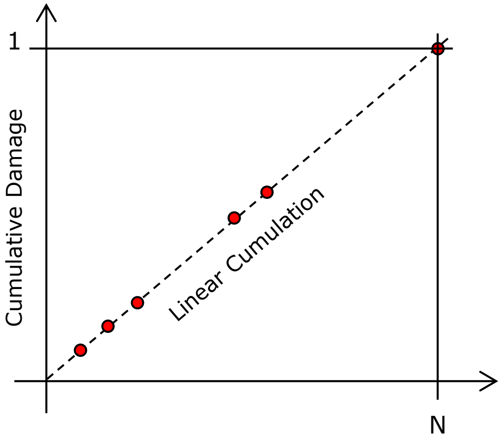
- Whatever the method is (Stress based or Strain based), damage for each cycle is obtained by entering in the modified curve (Basquin or Coffin-Manson) with stress amplitude (for stress based method) or strain amplitude (for strain based method) and extracting a number of cycles which represents the Life related to that specific cycle, i.e. how many of those cycles (constant amplitude) the component survives before a crack is nucleated.



DAMAGE CUMULATION RULE

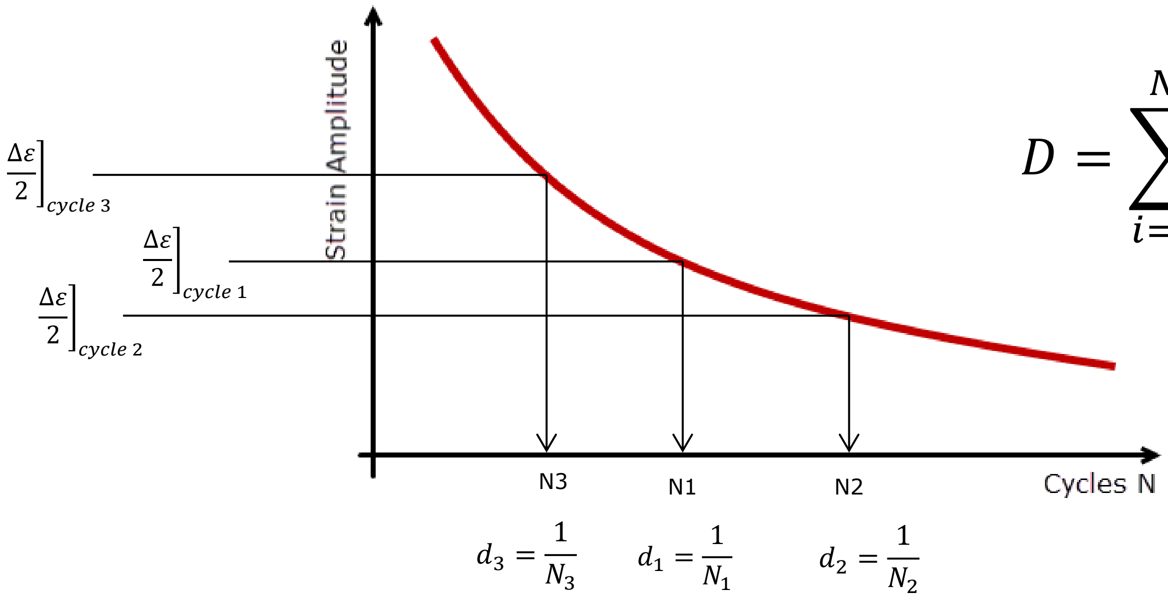
- Miner's Linear damage cumulation





DAMAGE CUMULATION RULE

- In case of variable amplitude sequences

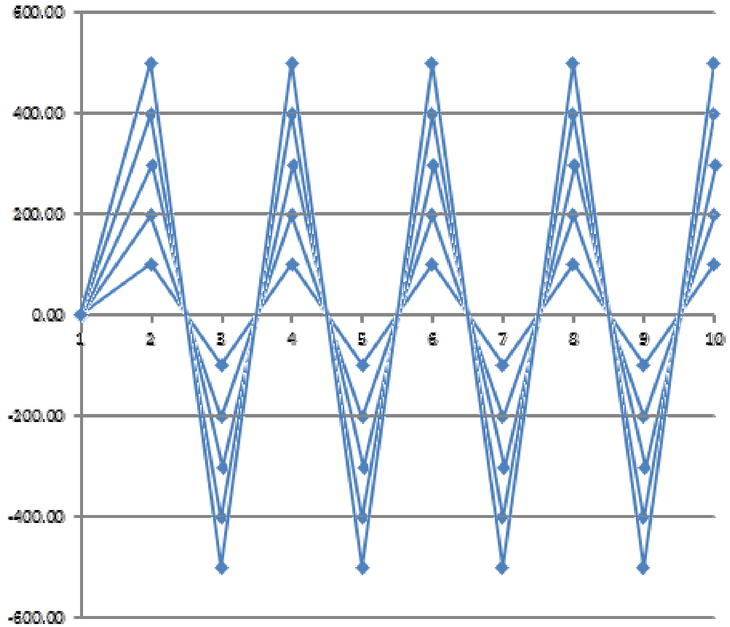


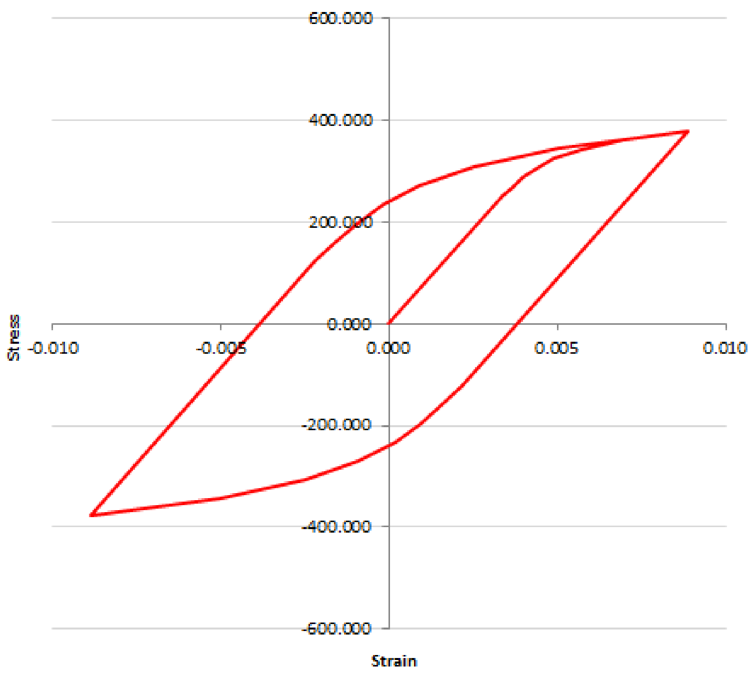
$$D = \sum_{i=1}^{N} d_i \qquad Life = \frac{T}{D}$$

(where *T* defines the sequence metric)

CYCLES DEFINITION – CYCLE COUNTING

- What is a cycle?
- How a stress cycle is defined?
- What is the physical meaning?





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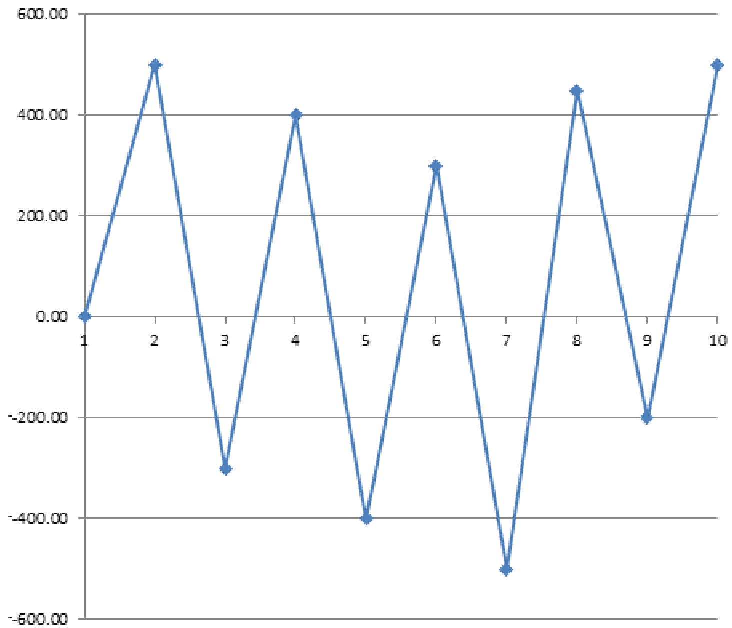
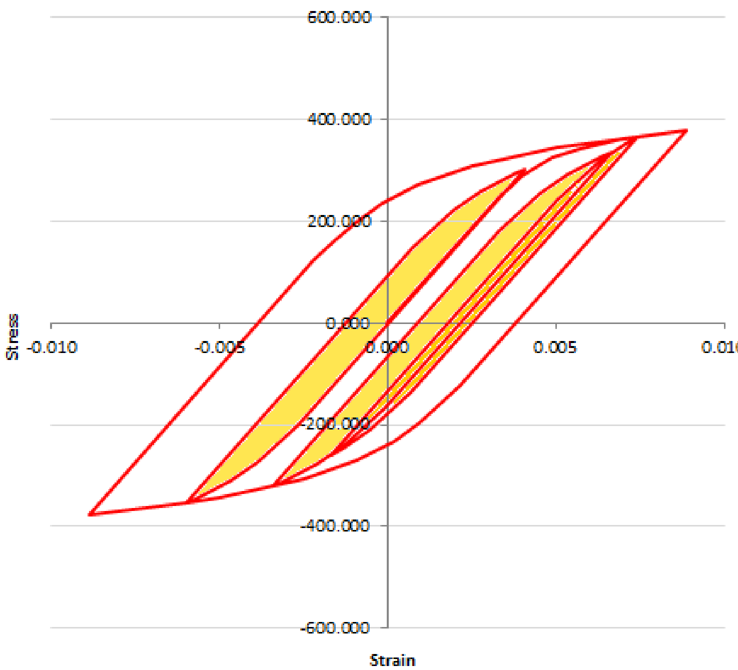
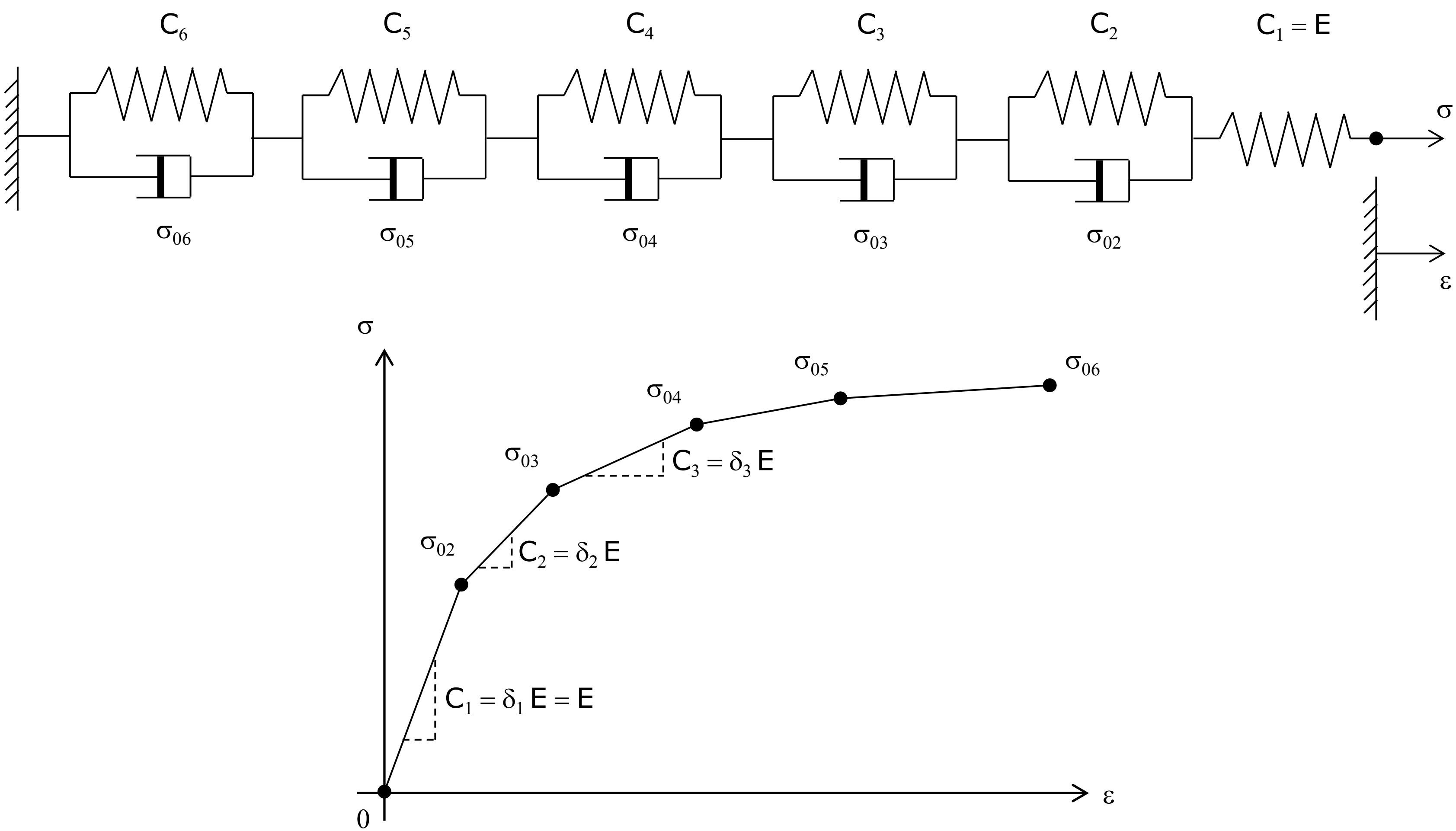


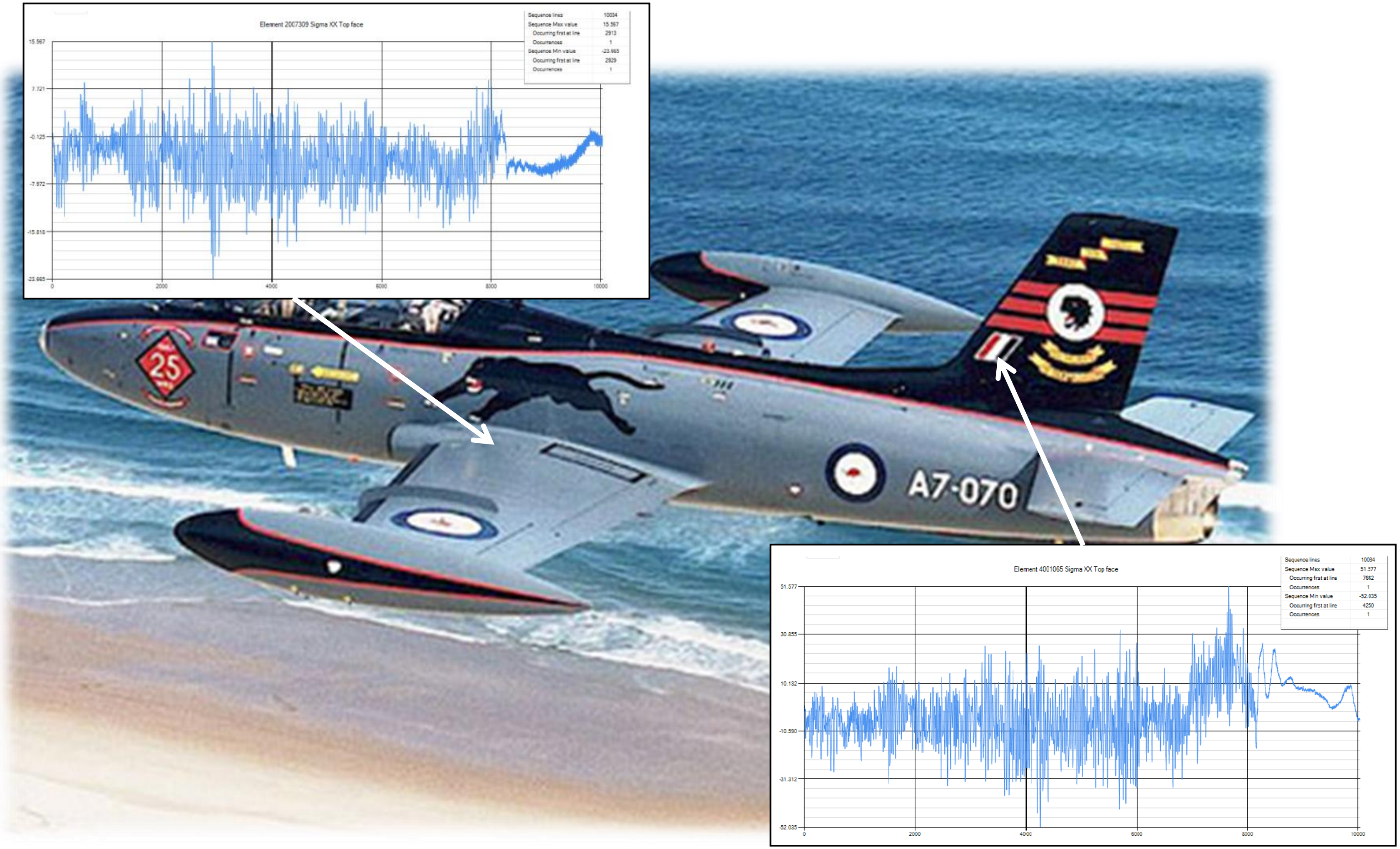


CYCLES DEFINITION – CYCLE COUNTING

- Rheological Model



CYCLES DEFINITION – CYCLE COUNTING



CYCLES DEFINITION – CYCLE COUNTING

- In order to avoid attacking the problem with an ‘incremental approach’ (which is computationally time consuming in case of long time histories), a method/tool is needed to extract cycles (i.e. closed loops) out of a Variable Amplitude spectrum.

- The most popular tool is the Rainflow Cycle Counting (accepted world-wide as the most appropriate for extracting stress/load cycles for fatigue analyses, the algorithm was developed by Endo and Matsuishi in 1968)

- In order to reduce computational time, normally signals are filtered (e.g. Racetrack Filter) before being counted: removal of non-turning points and ‘small’ cycles (i.e. negligibly damaging)

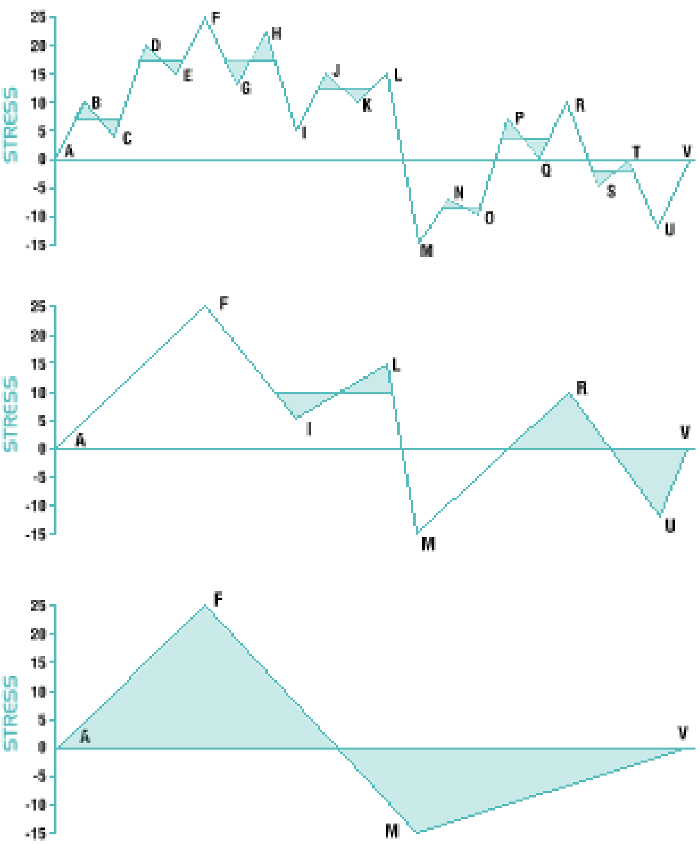
CYCLES DEFINITION – CYCLE COUNTING

- Two consecutive reversal points, *i* and *i-1*, within a sequence represent a peak and valley of a cycle if the conditions applies

$$S_{i-1} < \min(S_{i-2}, S_i)\,;\, S_i > max(S_{i-1}, S_{i+1})$$

or

$$S_{i-1} > \max(S_{i-2}, S_i)\,;\, S_i < min(S_{i-1}, S_{i+1})$$



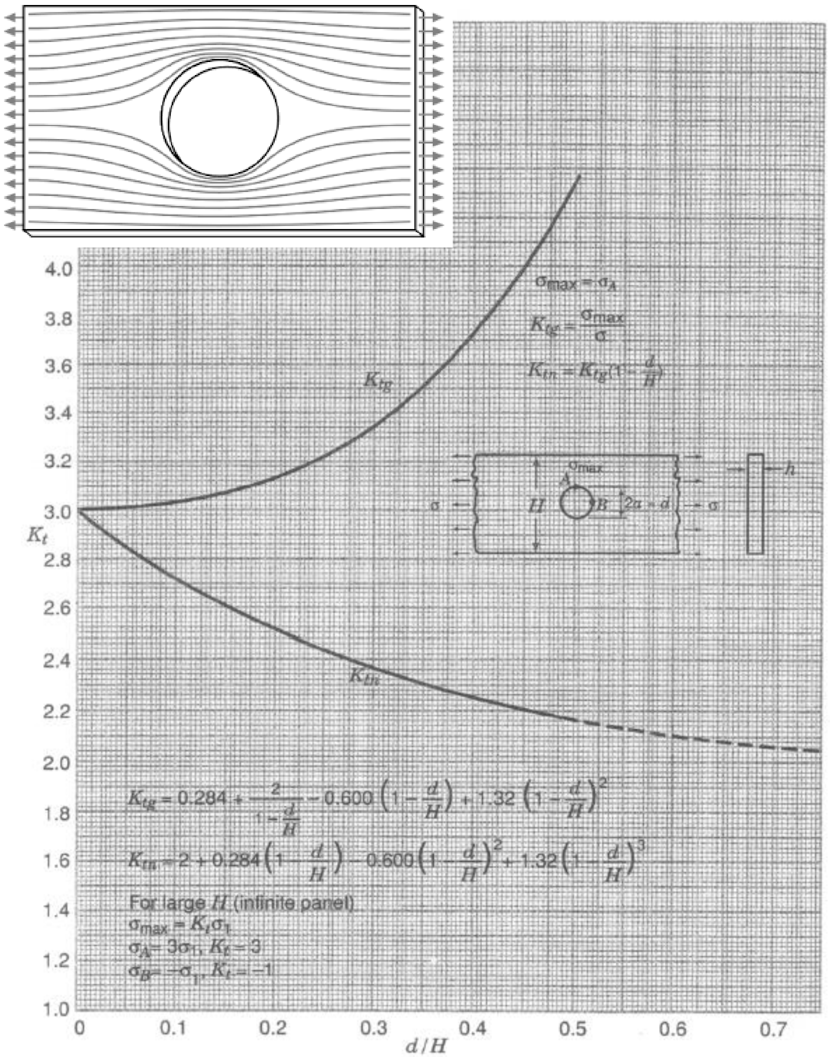
STRESS CONCENTRATION FACTORS

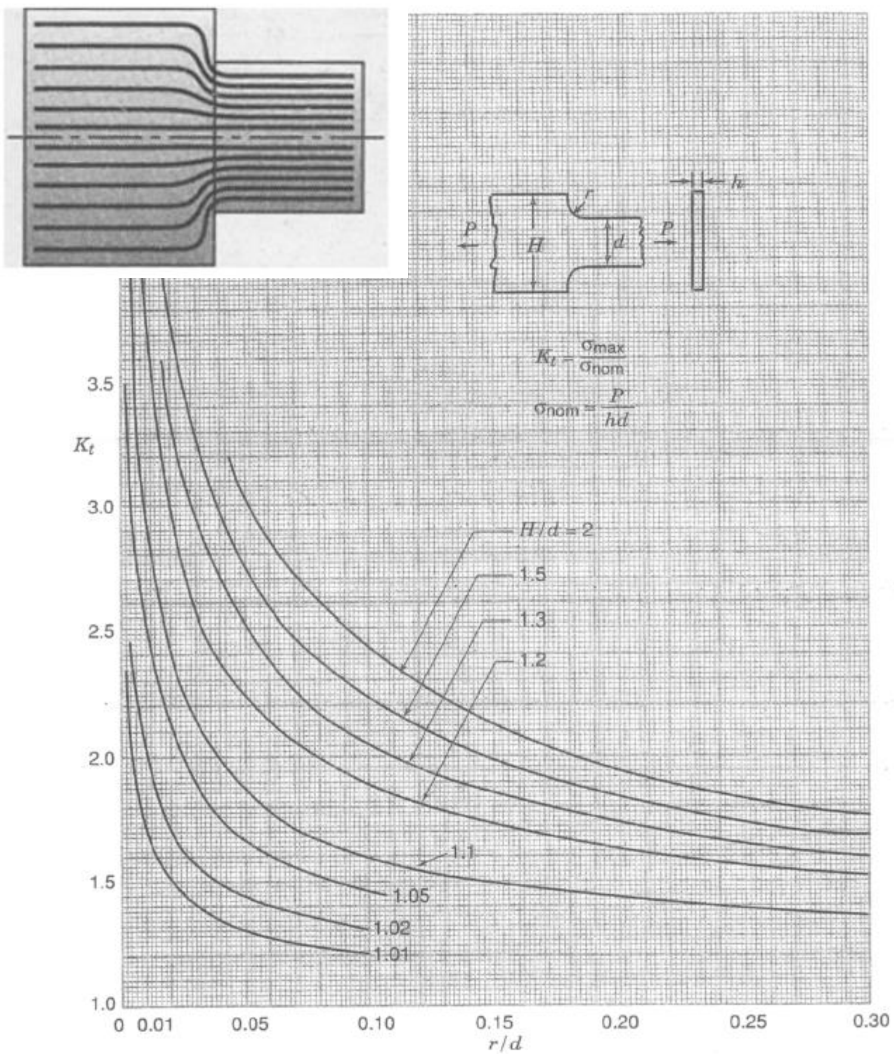
- Geometric discontinuities in a structure such as notches, holes, shoulders, grooves, … are details where stress concentrations occur (stress raisers).
- The Stress Concentration Factor *Kt* is the ratio between the local stress (maximum), at the stress raiser, and the far field (undisturbed), nominal stress

$$K_t = \frac{\sigma_{max}}{\sigma_{nominal}}$$

- Because of higher localized stresses, fatigue failure develops from such details.
- A collection of calculated stress concentration factors is provided by Peterson.

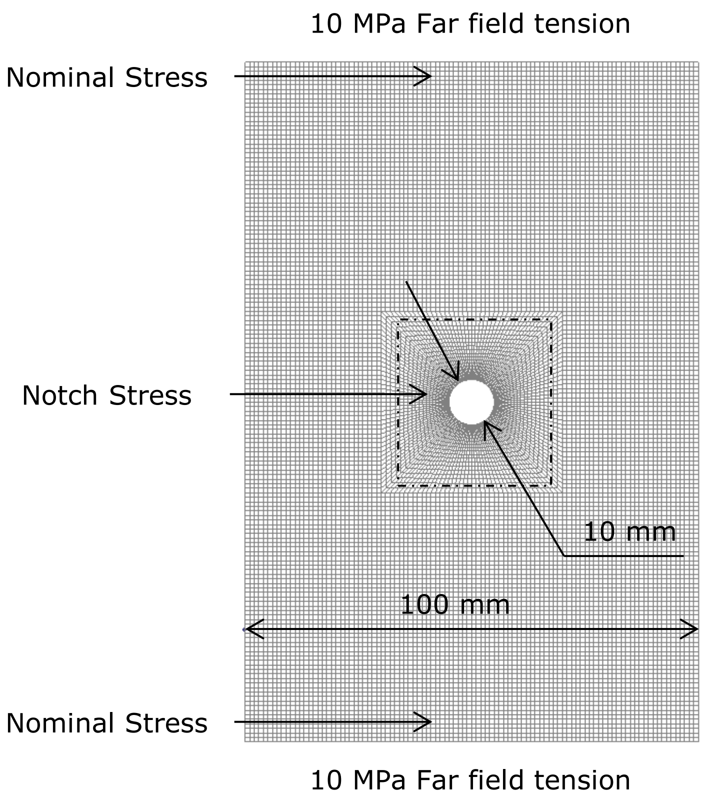
STRESS CONCENTRATION FACTORS

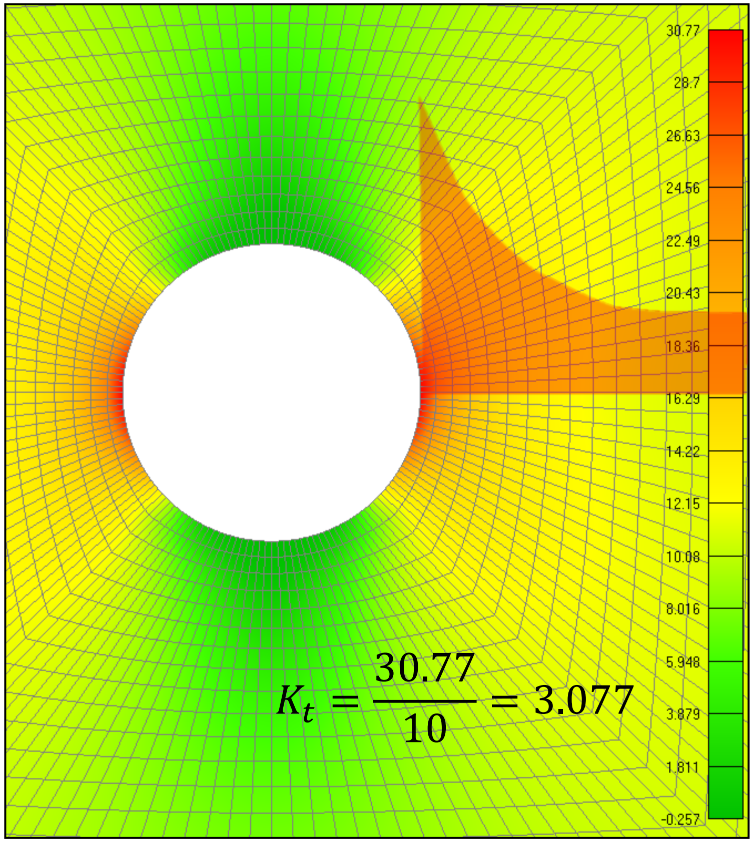




STRESS CONCENTRATION FACTORS

- Today, detailed Finite Element Models can be used to calculate numerically *Kt* for specific geometrical details

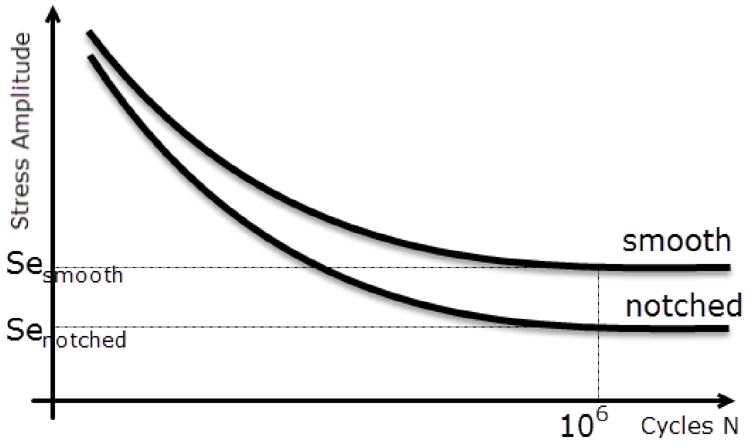




NOTCH FACTORS

- The use of theoretical *Kt*, coming from the assumption of ideal linear elastic materials, is not appropriate in case of applied alternating loads, i.e. fatigue.
- The use of Effective Stress Concentration Factors, or Notch Factors *Kf* is more appropriate in such cases.
- Defined as the Fatigue Strength ratio

$$K_f = \frac{Smooth\ Fatigue\ strength}{Notched\ Fatigue\ strength} = \frac{S_{e,smooth}}{S_{e,notched}}$$



- It is experimentally calculated (at long lives, i.e. $>10^6$ cycles).

NOTCH FACTORS

- K_f, differently from K_t, is not only geometry and load dependent, but also material dependent:

$$K_f = 1 + q(K_t - 1) = 1 + \frac{K_t - 1}{1 + {}^{a}/_{r}} \; < \; K_t$$

r = notch tip radius, a = material constant, q = notch sensitivity factor

- As a material length constant is involved, it implies that <u>two scaled geometries have same K_t but different K_f</u>
- For a given material, the smaller the notch is (small r), the smaller the notch sensitivity is

NOTCH FACTORS

Plate A

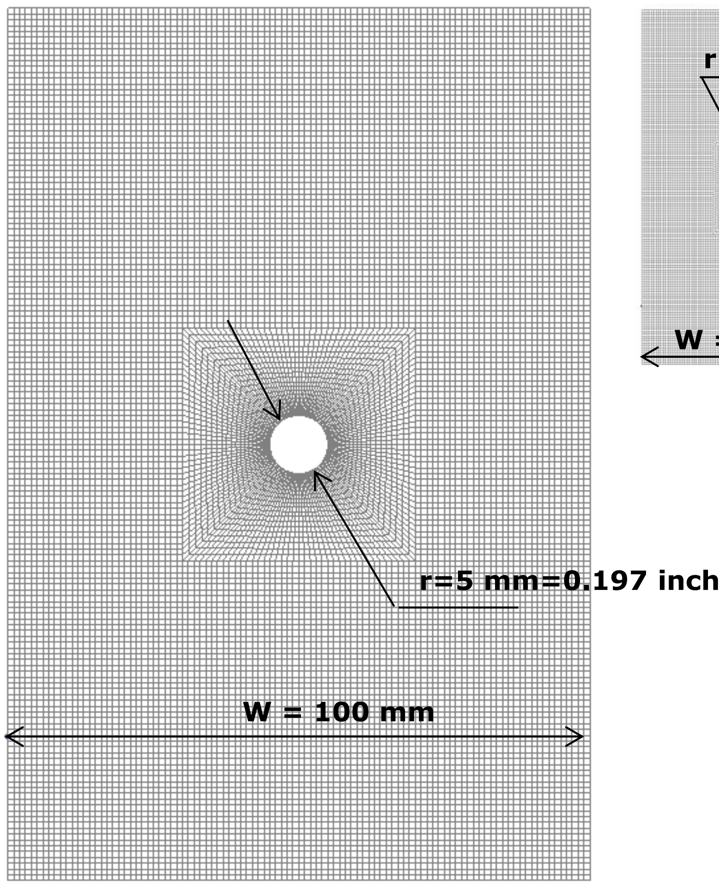
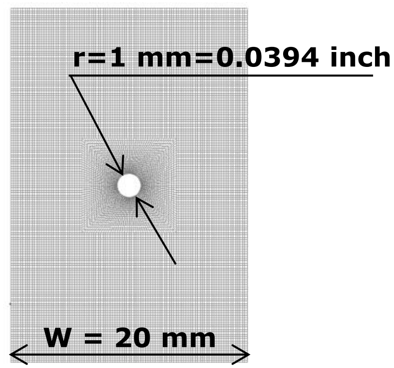


Plate B = 1/5 scaled from Plate A



- All dimensions of Plate B are scaled of 1/5 from Plate A BUT NOT the material dependent characteristic length a or ρ

Assume $a = 2.516 \cdot 10^{-2}\ inch$

For both plates $K_t = 3.077$ but:

For Plate A: $q_A = \frac{1}{1+{}^{a}/_{r}} = 0.8867 \rightarrow$

$$K_{fA} = 1 + q_A \cdot (K_t - 1) = 2.842$$

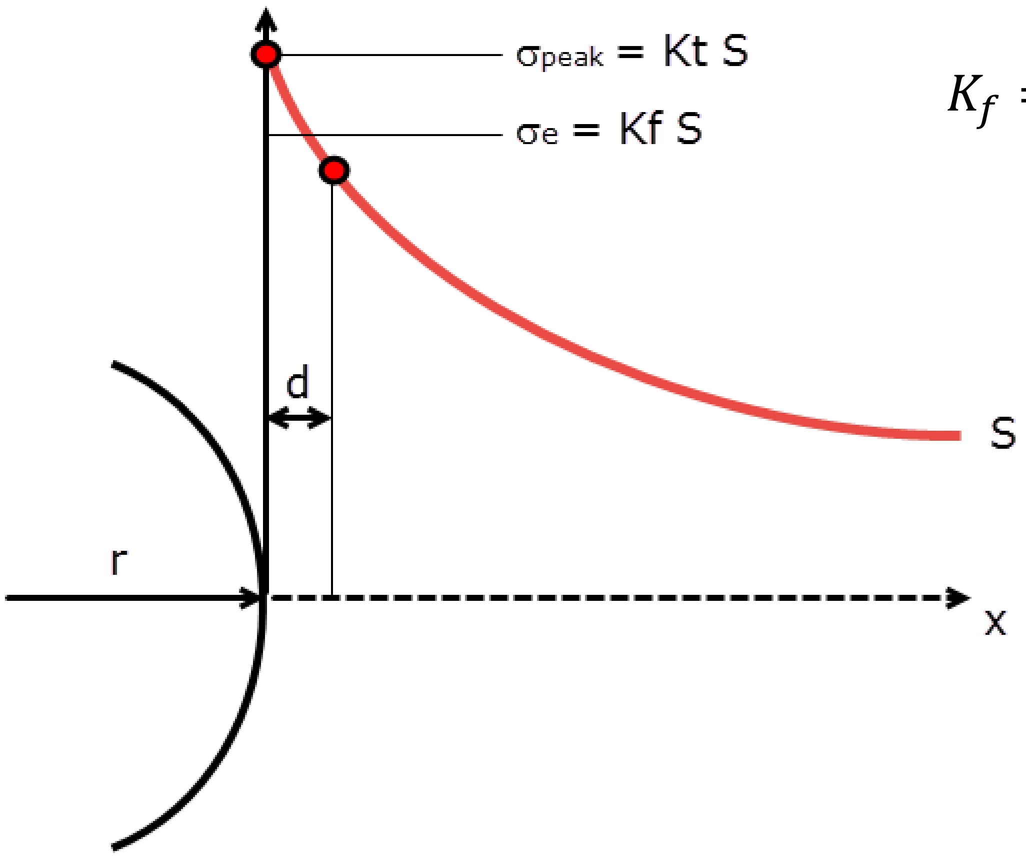
For Plate B: $q_B = \frac{1}{1+{}^{a}/_{r}} = 0.6103 \rightarrow$

$$K_{fB} = 1 + q_B \cdot (K_t - 1) = 2.267$$

- It derives that Plate B has higher Fatigue Strength w.r.t. Plate A

NOTCH FACTORS

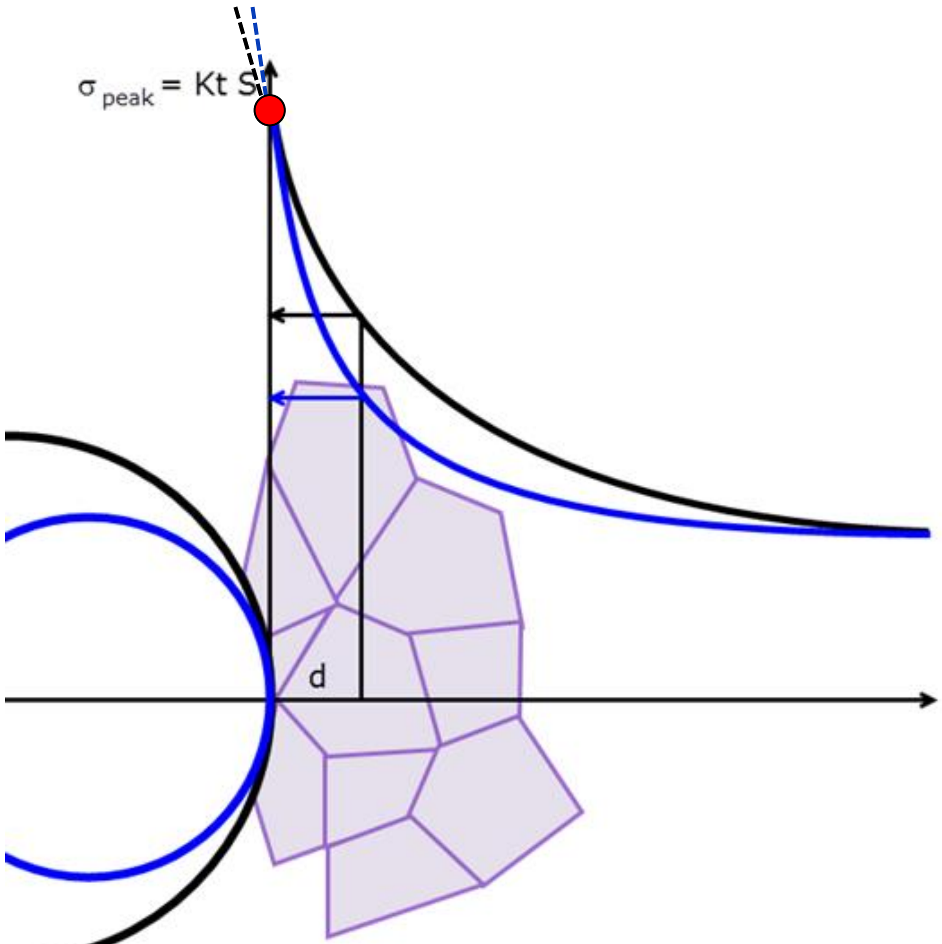
- A way to interpret the notch factor (from Dowling).



$$K_f = \frac{Stress\ average\ out\ to\ x = d}{S} = \frac{\sigma_e}{S}$$

- The stress controlling initiation of fatigue damage IS NOT the highest stress at the notch surface (x=0), but rather the somewhat lower value that is average out to a distance x=d.

NOTCH FACTORS



- Scaled geometries: macroscopic dimensions are scaled; the material characteristic dimensions (e.g. grain size) are not scaled.
- The 'critical distance' d stays constant and this drives the difference in resulting Notch factors.
- Some methods focus on the critical distance, some others (e.g. FKM) focus on the stress gradient at the notch.

LOCAL ELASTIC-PLASTIC STRESS STATUS CALCULATION

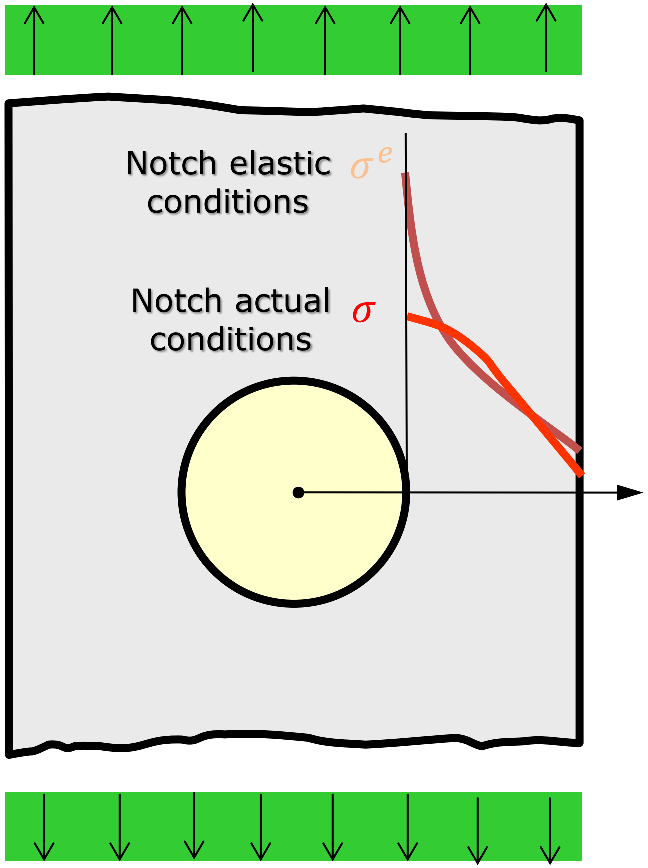
- A key step within the Strain-based approach is the determination of local elastic-plastic stresses occurring at the notch.

- We calculate, with a FEM, generally the elastic stresses and strains

$$\sigma^e \text{ and } \varepsilon^e$$

- We MUST calculate the elastic-plastic stresses and strains

$$\sigma \text{ and } \varepsilon$$



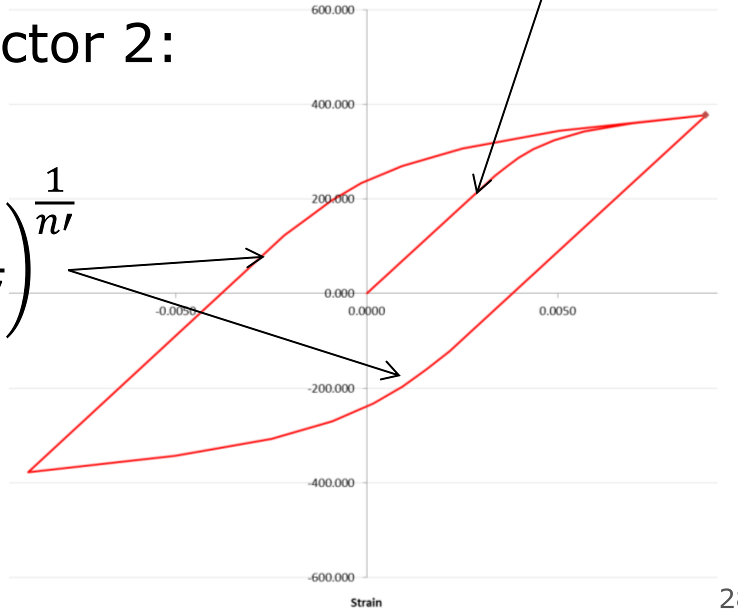
LOCAL ELASTIC-PLASTIC STRESS STATUS CALCULATION

- The cyclic Stress-Strain curve can be modelled by means of the Ramberg-Osgood equation:

$$\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{K'}\right)^{\frac{1}{n'}}$$

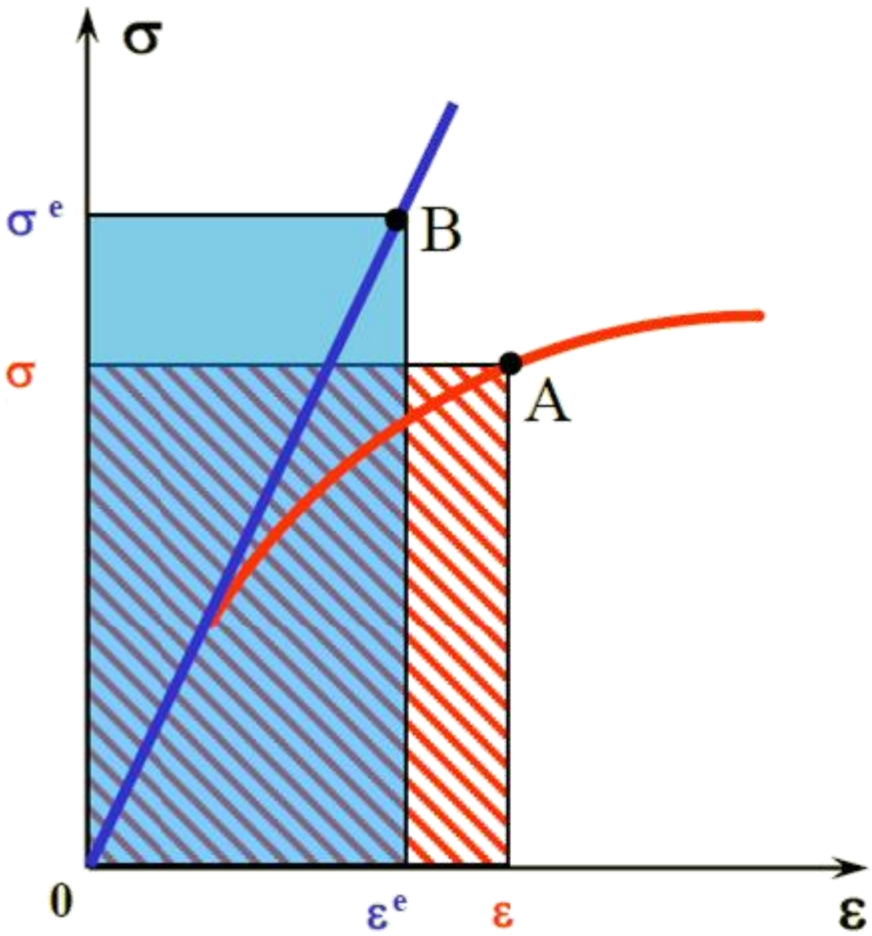
- The equation for the Hysteresis Loops retains the same Ramberg-Osgood structure, with a factor 2:

$$\frac{\Delta\varepsilon}{2} = \frac{\Delta\sigma}{2E} + \left(\frac{\Delta\sigma}{2K'}\right)^{\frac{1}{n'}}$$



LOCAL ELASTIC-PLASTIC STRESS STATUS CALCULATION

- Neuber approach



$$\sigma^e \cdot \varepsilon^e = \sigma \cdot \varepsilon$$

or in 'hysteresis format'

$$\begin{cases} \Delta\sigma^e \cdot \Delta\varepsilon^e = \Delta\sigma \cdot \Delta\varepsilon \\ \dfrac{\Delta\varepsilon}{2} = \dfrac{\Delta\sigma}{2E} + \left(\dfrac{\Delta\sigma}{2K'}\right)^{\frac{1}{n'}} \end{cases}$$

2 equations, 2 unknows: $\Delta\sigma, \Delta\varepsilon$

(being $\sigma^e \cdot \varepsilon^e$ or $\Delta\sigma^e \cdot \Delta\varepsilon^e$ known from the FEM solution)

LOCAL ELASTIC-PLASTIC STRESS STATUS CALCULATION

- Neuber approach

$$\Delta\sigma^e \cdot \Delta\varepsilon^e = \Delta\sigma \cdot \Delta\varepsilon$$

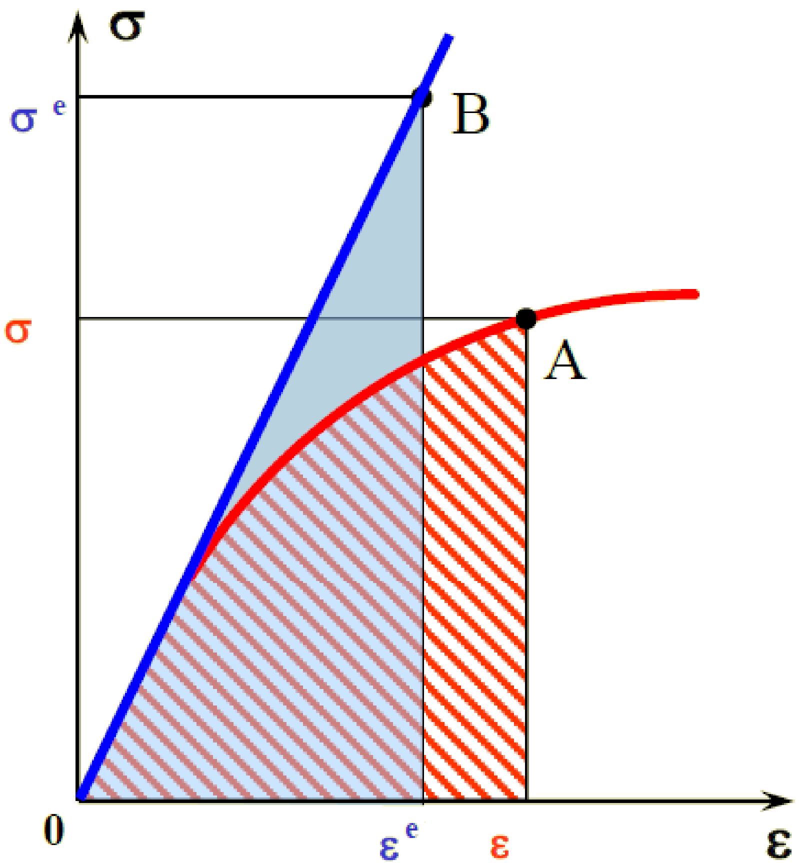
$$\Delta\sigma^e \cdot \Delta\varepsilon^e = \Delta\sigma^e \cdot \frac{\Delta\sigma^e}{E} = \frac{(\Delta\sigma^e)^2}{E}$$

$$\Delta\sigma \cdot \Delta\varepsilon = \Delta\sigma \cdot 2\left(\frac{\Delta\sigma}{2E} + \left(\frac{\Delta\sigma}{2K'}\right)^{\frac{1}{n'}}\right) = \frac{\Delta\sigma^2}{E} + 2\Delta\sigma\left(\frac{\Delta\sigma}{2K'}\right)^{\frac{1}{n'}}$$

$$\frac{(\Delta\sigma^e)^2}{E} = \frac{\Delta\sigma^2}{E} + 2\Delta\sigma\left(\frac{\Delta\sigma}{2K'}\right)^{\frac{1}{n'}}$$

LOCAL ELASTIC-PLASTIC STRESS STATUS CALCULATION

- E.S.E.D. Glinka approach



$$\int_0^{\varepsilon^e} \sigma^e \cdot d\varepsilon^e = \int_0^{\varepsilon} \sigma \cdot d\varepsilon$$

or in 'hysteresis format'

$$\begin{cases} \int_0^{\Delta\varepsilon^e} \Delta\sigma^e \cdot d\Delta\varepsilon^e = \int_0^{\Delta\varepsilon} \Delta\sigma \cdot d\Delta\varepsilon \\ \frac{\Delta\varepsilon}{2} = \frac{\Delta\sigma}{2E} + \left(\frac{\Delta\sigma}{2K'}\right)^{\frac{1}{n'}} \end{cases}$$

2 equations, 2 unknows: $\Delta\sigma, \Delta\varepsilon$

(being $\int_0^{\Delta\varepsilon^e} \Delta\sigma^e \cdot d\Delta\varepsilon^e$ known from the FEM solution)

LOCAL ELASTIC-PLASTIC STRESS STATUS CALCULATION

- E.S.E.D. Glinka approach

$$\int_0^{\Delta\varepsilon^e} \Delta\sigma^e \cdot d\Delta\varepsilon^e \quad = \quad \int_0^{\Delta\varepsilon} \Delta\sigma \cdot d\Delta\varepsilon$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\int_0^{\Delta\varepsilon^e} \Delta\sigma^e \cdot d\Delta\varepsilon^e = \int_0^{\Delta\sigma^e} \Delta\sigma^e \cdot d\frac{\Delta\sigma^e}{E} = \frac{(\Delta\sigma^e)^2}{2E}$$

$$\int_0^{\Delta\varepsilon} \Delta\sigma \cdot d\Delta\varepsilon = 2\int_0^{\Delta\sigma} \Delta\sigma \cdot d\left[\frac{\Delta\sigma}{2E} + \left(\frac{\Delta\sigma}{2K'}\right)^{\frac{1}{n'}}\right] = \frac{\Delta\sigma^2}{2E} + \frac{2\Delta\sigma}{n'+1}\left(\frac{\Delta\sigma}{2K'}\right)^{\frac{1}{n'}}$$

$$\frac{(\Delta\sigma^e)^2}{2E} = \frac{\Delta\sigma^2}{2E} + \frac{2\Delta\sigma}{n'+1}\left(\frac{\Delta\sigma}{2K'}\right)^{\frac{1}{n'}}$$

LOCAL ELASTIC-PLASTIC STRESS STATUS CALCULATION

- Neuber approach

$$\frac{(\Delta\sigma^e)^2}{E} = \frac{\Delta\sigma^2}{E} + 2\Delta\sigma\left(\frac{\Delta\sigma}{2K'}\right)^{\frac{1}{n'}}$$

- E.S.E.D. Glinka approach

$$\frac{(\Delta\sigma^e)^2}{E} = \frac{\Delta\sigma^2}{E} + \frac{4\Delta\sigma}{n'+1}\left(\frac{\Delta\sigma}{2K'}\right)^{\frac{1}{n'}}$$

- The two equations differ for the factor 2/(n'+1) in the Glinka approach form. Since 0<n'<1 → 2/(n'+1)<1. Smaller notch stress (and therefore smaller notch strain) is predicted with the Glinka approach, resulting in longer fatigue life compared to the Neuber's rule.

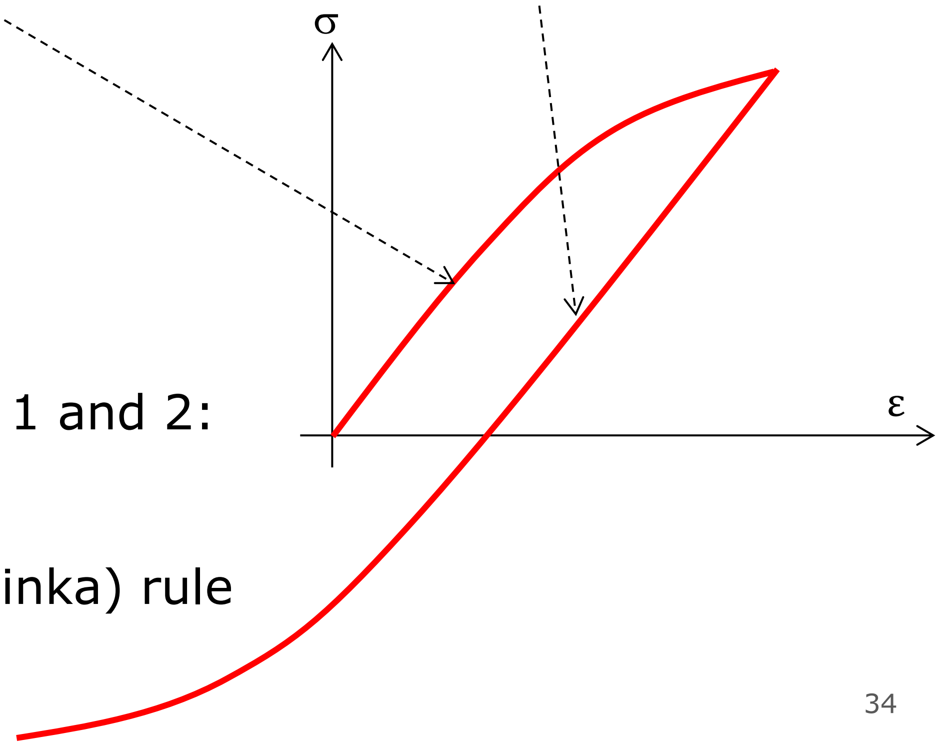
LOCAL ELASTIC-PLASTIC STRESS STATUS CALCULATION

- Generalizing:

$$\frac{(\sigma^e)^2}{E} = \frac{\sigma^2}{E} + \bar{\alpha}_U \sigma \left(\frac{\sigma}{K'}\right)^{\frac{1}{n'}} \qquad \frac{(\Delta\sigma^e)^2}{E} = \frac{\Delta\sigma^2}{E} + 2\bar{\alpha}_U \Delta\sigma \left(\frac{\Delta\sigma}{2K'}\right)^{\frac{1}{n'}}$$

monotonic

cyclic



where

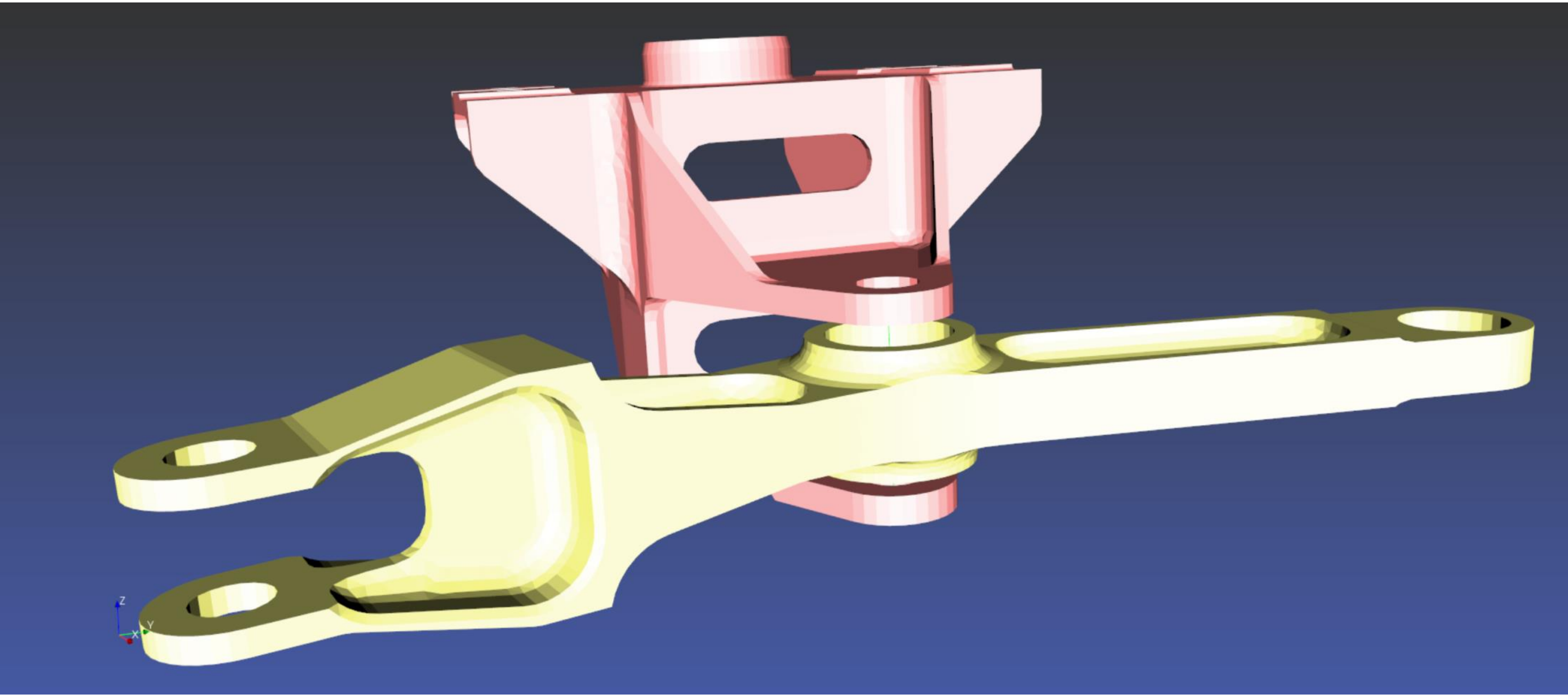
$$\bar{\alpha}_U = \frac{\alpha_U + n' \cdot (2 - \alpha_U)}{1 + n'}$$

The parameter α_U spaninng between 1 and 2:

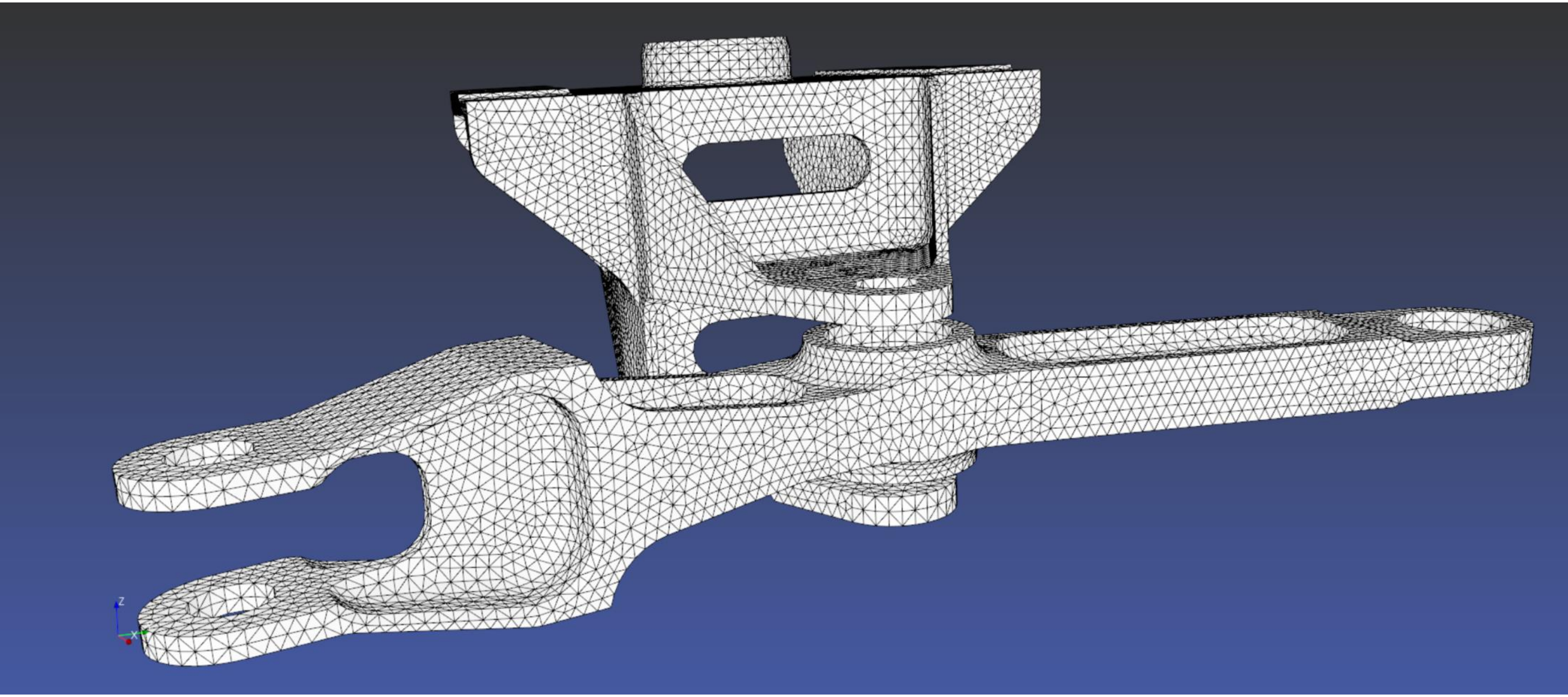
$\alpha_U = 1 \rightarrow \bar{\alpha}_U = 1 \rightarrow$ Neuber rule

$\alpha_U = 2 \rightarrow \bar{\alpha}_U = \frac{2}{1+n'} \rightarrow$ E.S.E.D. (Glinka) rule

MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS



MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS



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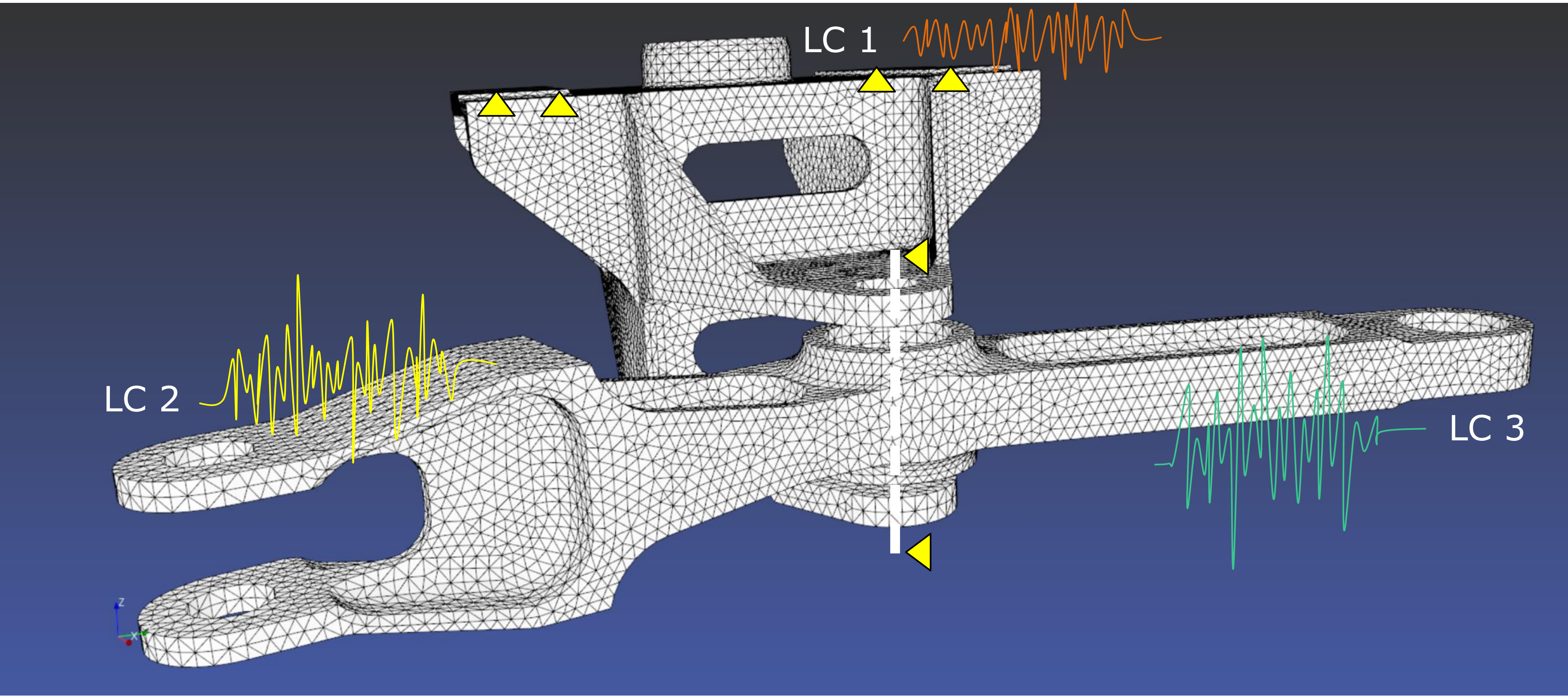

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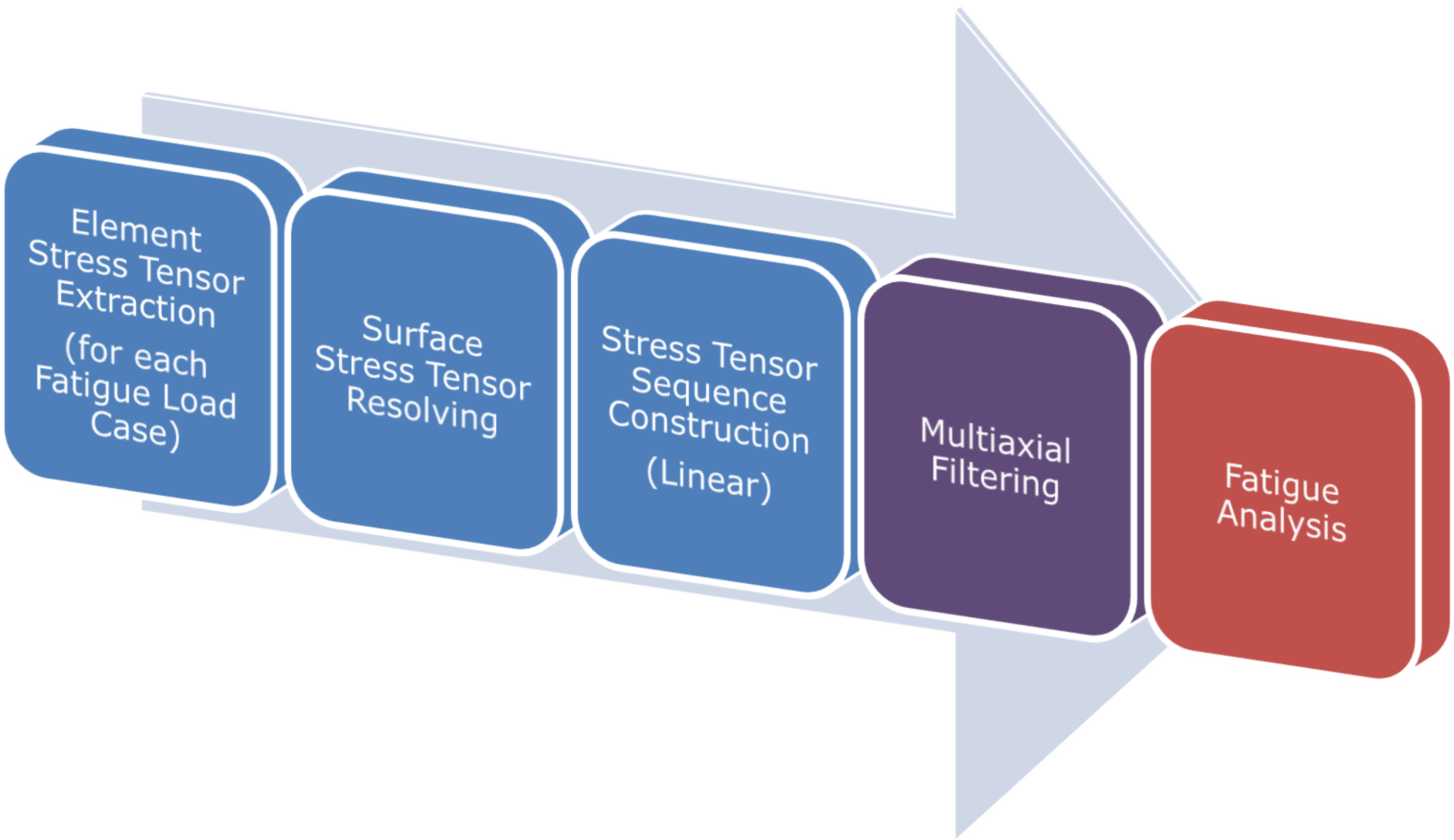
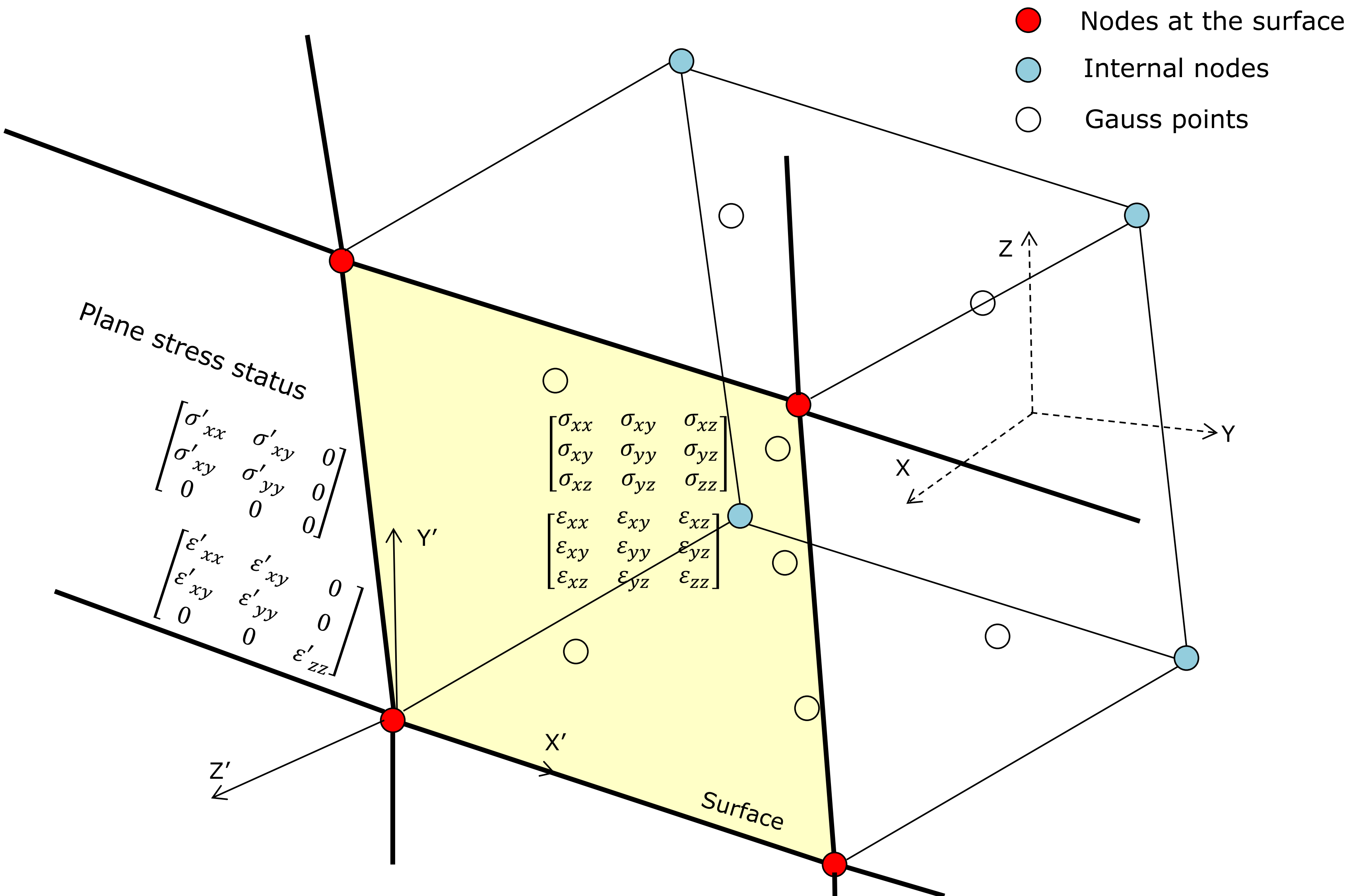
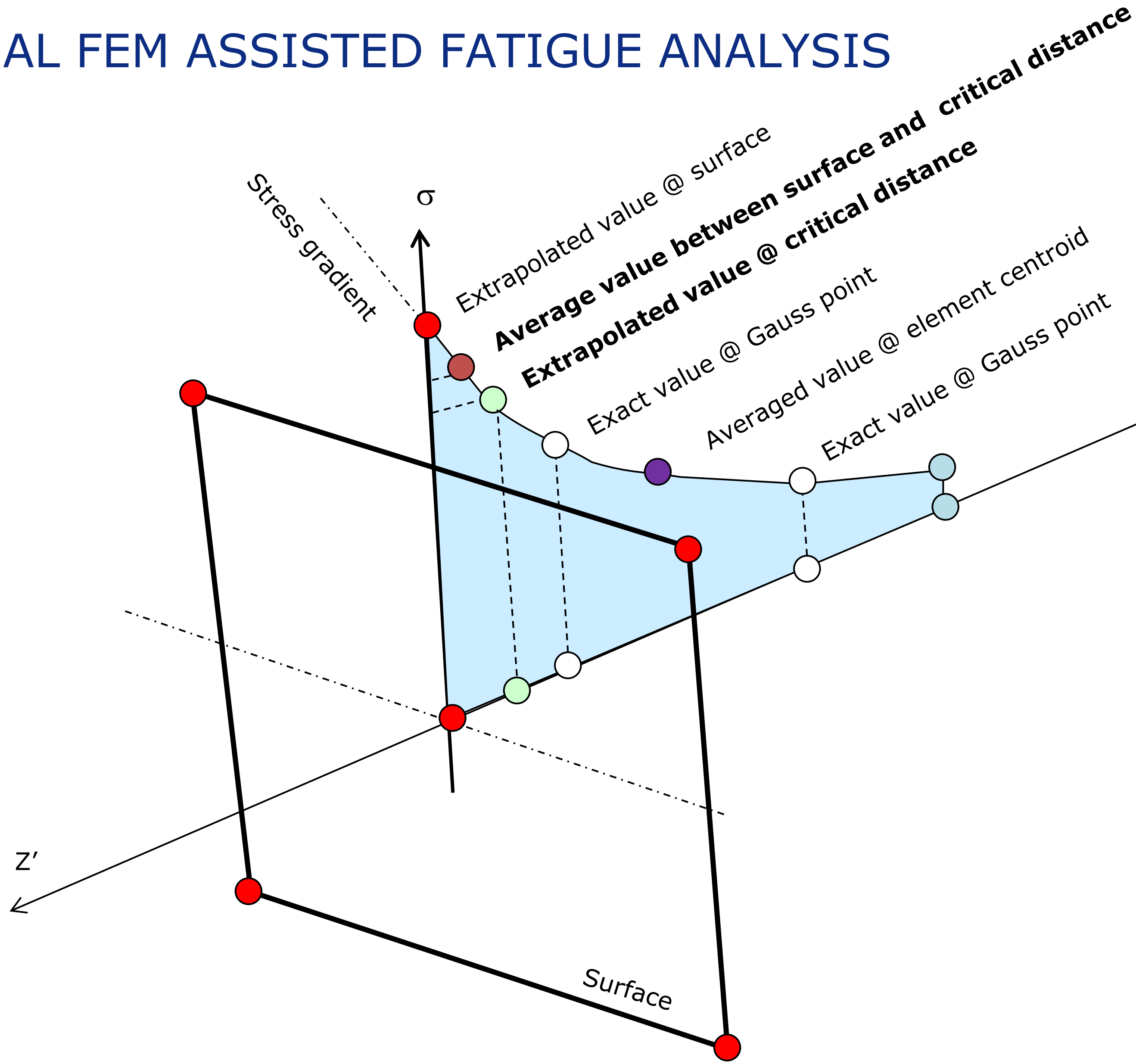


MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS



MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

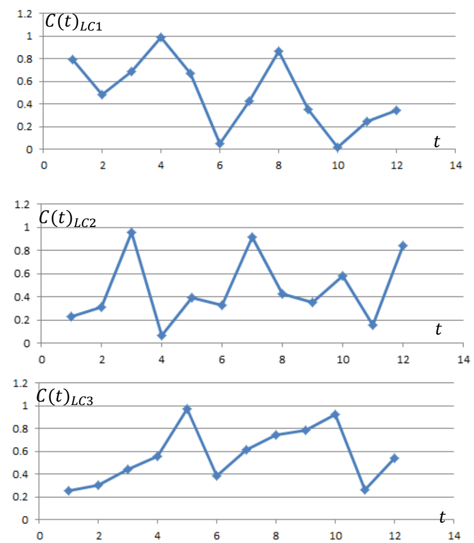


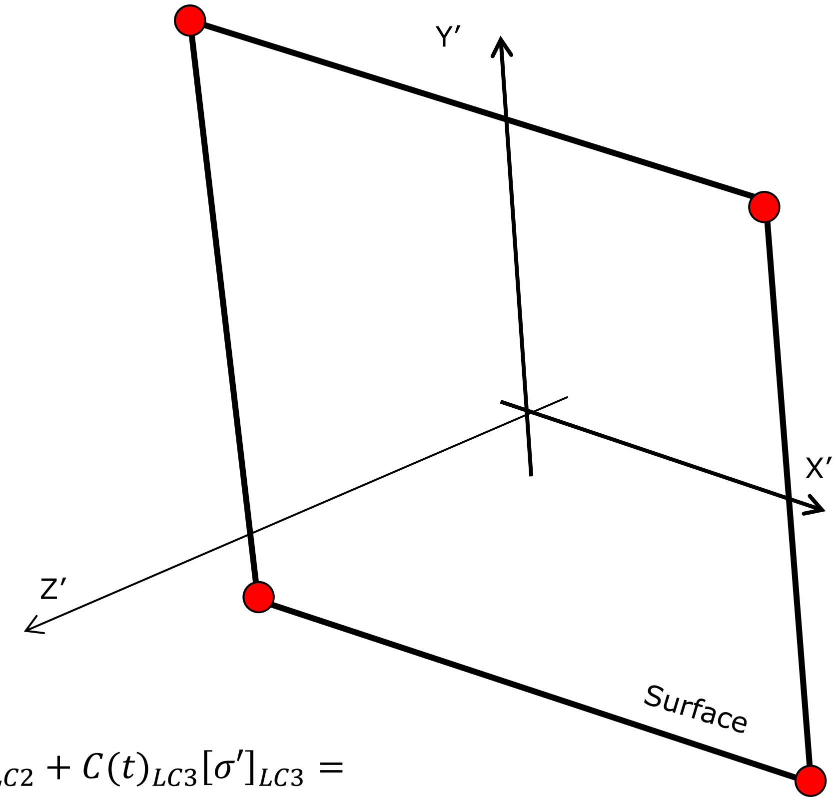
MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

$$[\sigma']_{LC1} = \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{xy} & \sigma'_{yy} \end{bmatrix}_{LC1}$$

$$[\sigma']_{LC2} = \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{xy} & \sigma'_{yy} \end{bmatrix}_{LC2}$$

$$[\sigma']_{LC3} = \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{xy} & \sigma'_{yy} \end{bmatrix}_{LC3}$$





$$[\sigma'(t)]_{tot} = \begin{bmatrix} \sigma'_{xx}(t) & \sigma'_{xy}(t) \\ \sigma'_{xy}(t) & \sigma'_{yy}(t) \end{bmatrix} = C(t)_{LC1}[\sigma']_{LC1} + C(t)_{LC2}[\sigma']_{LC2} + C(t)_{LC3}[\sigma']_{LC3} =$$

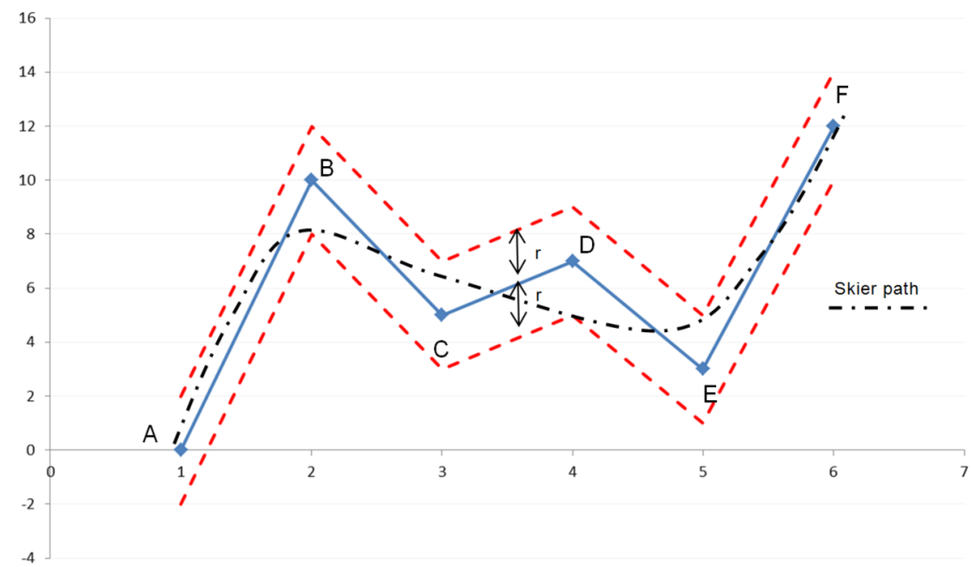
$$= C(t)_{LC1} \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{xy} & \sigma'_{yy} \end{bmatrix}_{LC1} + C(t)_{LC2} \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{xy} & \sigma'_{yy} \end{bmatrix}_{LC2} + C(t)_{LC3} \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{xy} & \sigma'_{yy} \end{bmatrix}_{LC3} = \sum_i C(t)_{LCi}[\sigma']_{LCi}$$

MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

- Multiaxial Filtering – Racetrack Filter
  - It is derived from uniaxial racetrack filter

Originally inspired by slalom ski races, this amplitude filter idea involves drawing a ‘racetrack’ of width 2r bounded by upper and lower fences that have the same profile of the original sequence.

Every time a skier racing in this racetrack needs to change direction, a reversal point is identified.



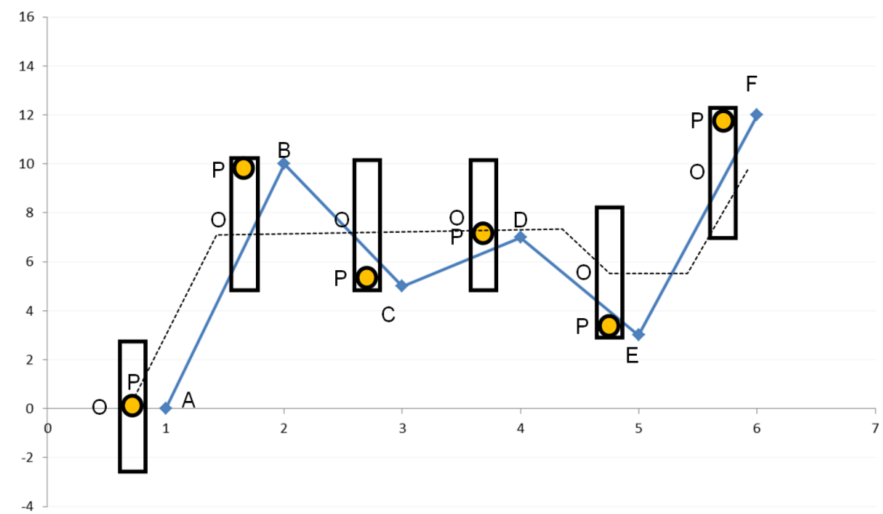
Clearly, the track width 2r determines the number of points which will be eliminated and the number of reversals which remain: wider tracks filter out most of the original sequence points, while narrow tracks almost keep all the original reversals.

MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

- Multiaxial Filtering – Racetrack Filter
  - It is derived from uniaxial racetrack filter

The algorithm of the racetrack filter is easier implemented through another physical analogy: a small peg P oscillating inside a slotted plate whose center is the point O.

The range that the peg can oscillate inside the slot is 2r. both the peg and the slot are initially centered with the point A.



During the path AB, the peg moves up until reaching the upper limit of the slotted plate, which then starts moving up.

It is seen that the path BCD does not involve any translation of the slotted plate, meaning that points C and D can be filtered out. Then both the paths DE and EF involve translation of the slotted plate.

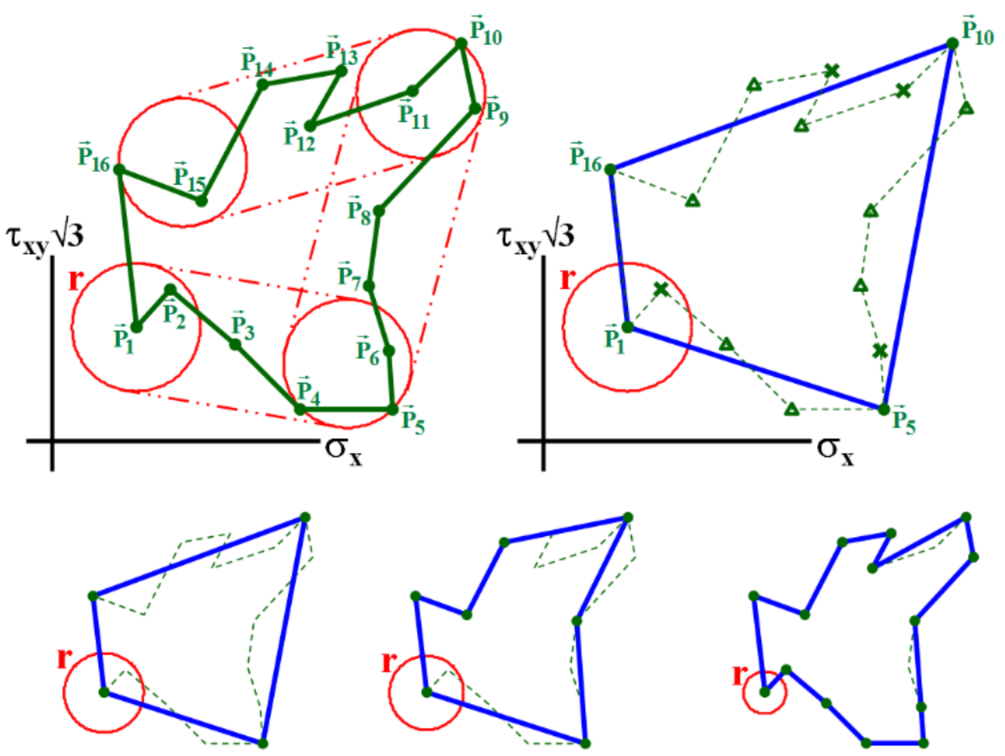
MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

- Multiaxial Filtering – Racetrack Filter
  - Extension to Multiaxial time history

An hyper-sphere in the deviatoric space is defined. The radius defines the 'resolution' of the filter.

The stress status defines the peg in the hyper-sphere. When the peg reaches the hyper-sphere surface and tries to move out of it, both the peg and the hyper-sphere translate altogether.

The filtered history is composed by the **initial point** and **all kinking and reversal points**.



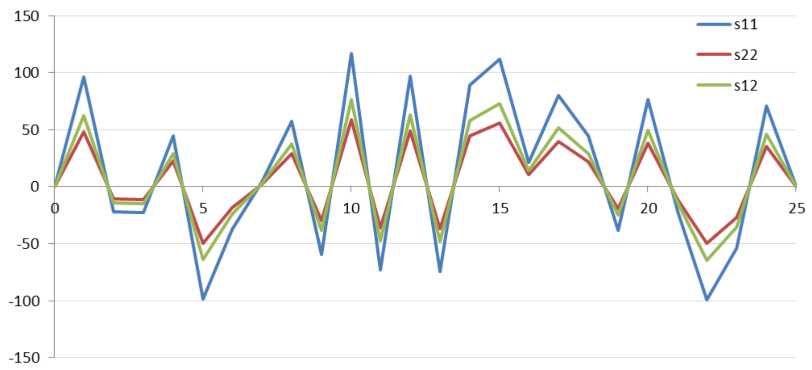
MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

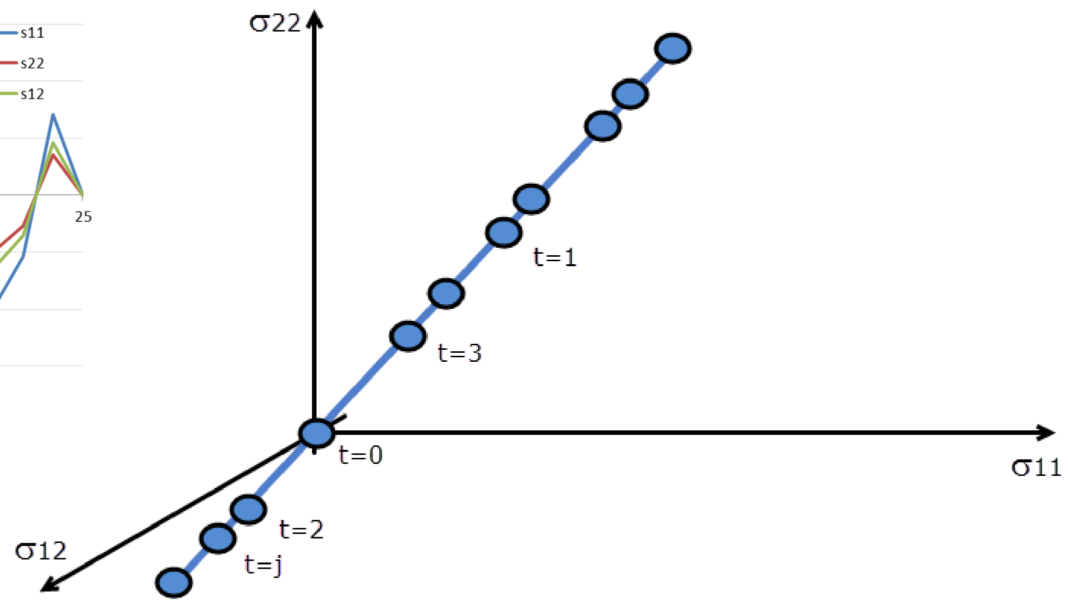
- Models and methods described in Part 1 relate to UNIAXIAL conditions
  - One stress or one strain (with its own time history),
  - The fatigue parameters is built with one stress or one strain

- Dealing with Multiaxial Stress Tensors, the Fatigue analysis problem gets significantly more complex.
- Depending on the nature of the applied loads, the Multiaxial problems are divided into two categories:
  - Multiaxial – Proportional Loadings
  - Multiaxial – Non-Proportional Loadings

MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

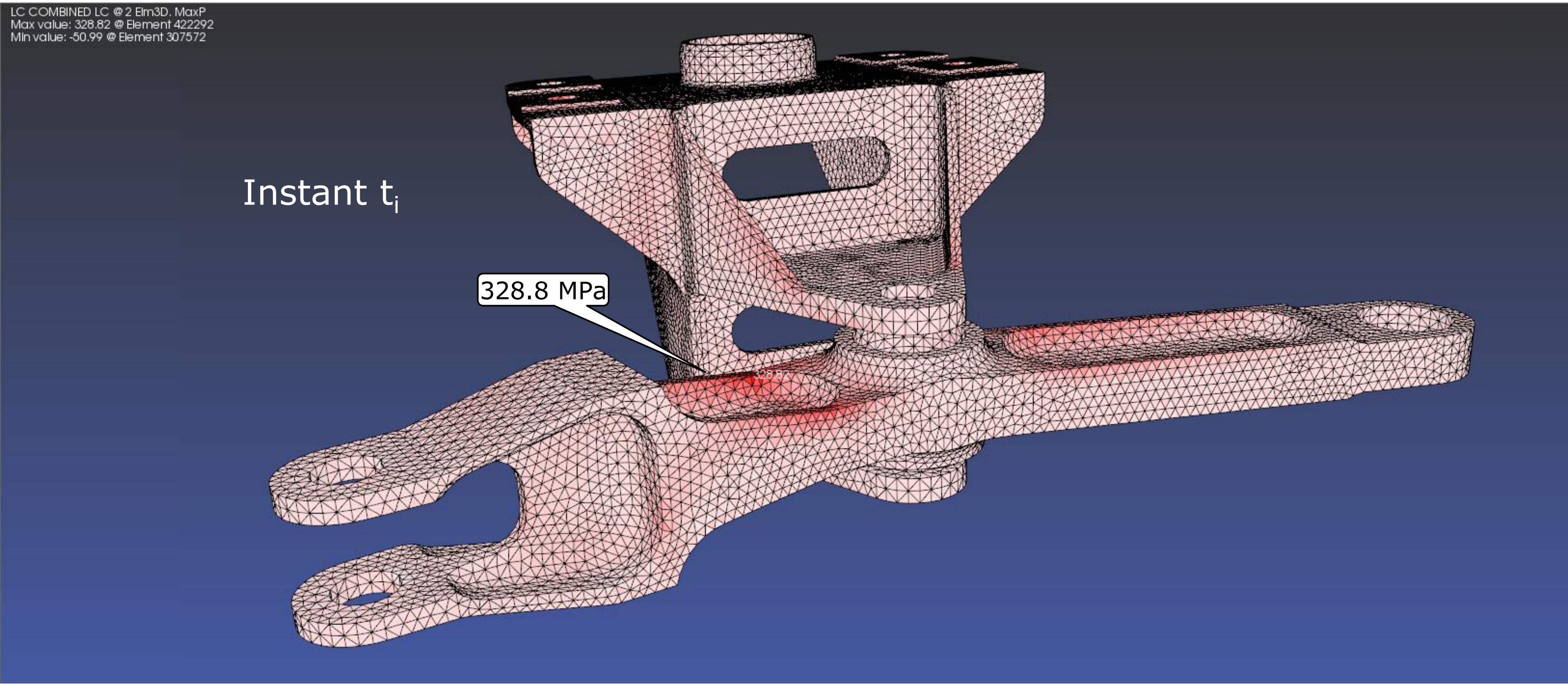
- Multiaxial – Proportional Loadings.
  - This situation typically occurs when the structure is subjected to a single load, whose magnitude changes over time, or when the structure is subjected to a set of loads which change all in phase over time





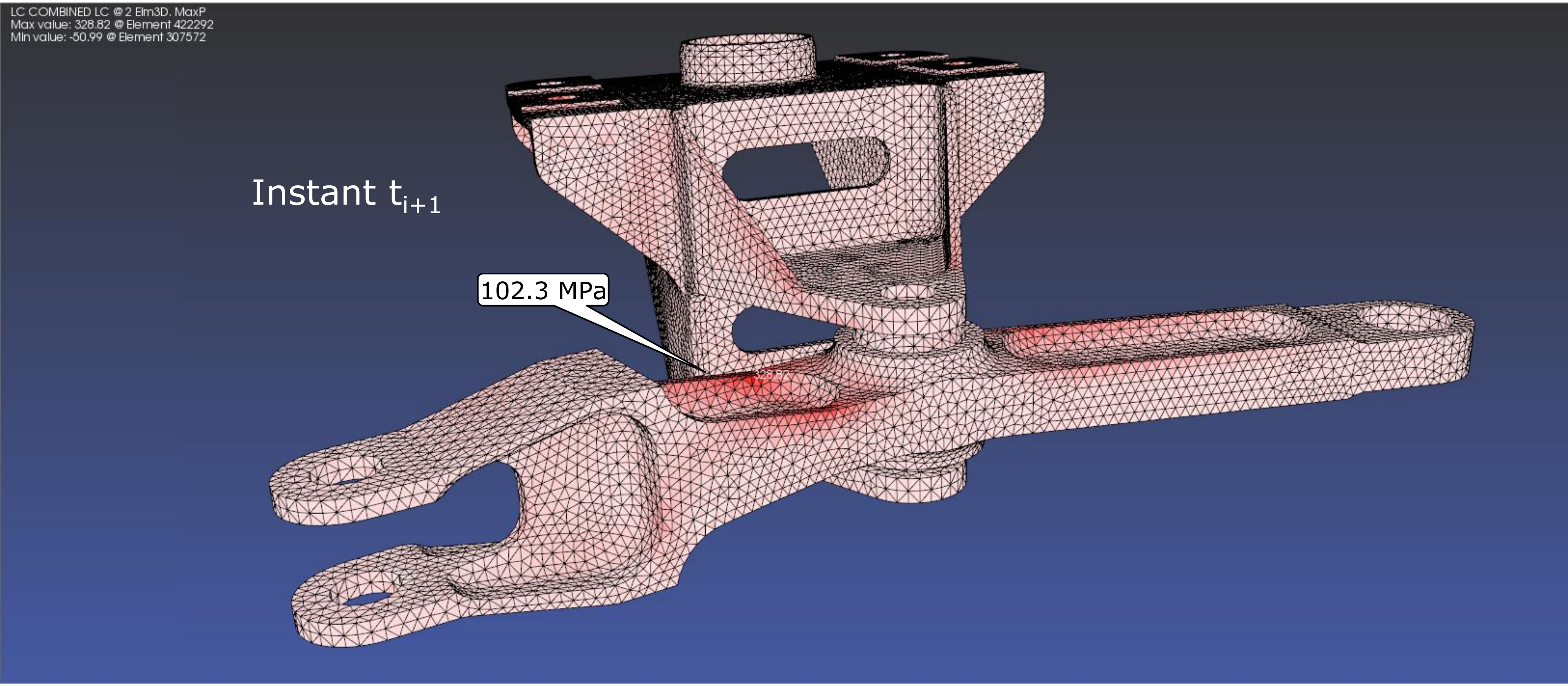
MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

- Multiaxial Proportional Loading Conditions (software LIFING)



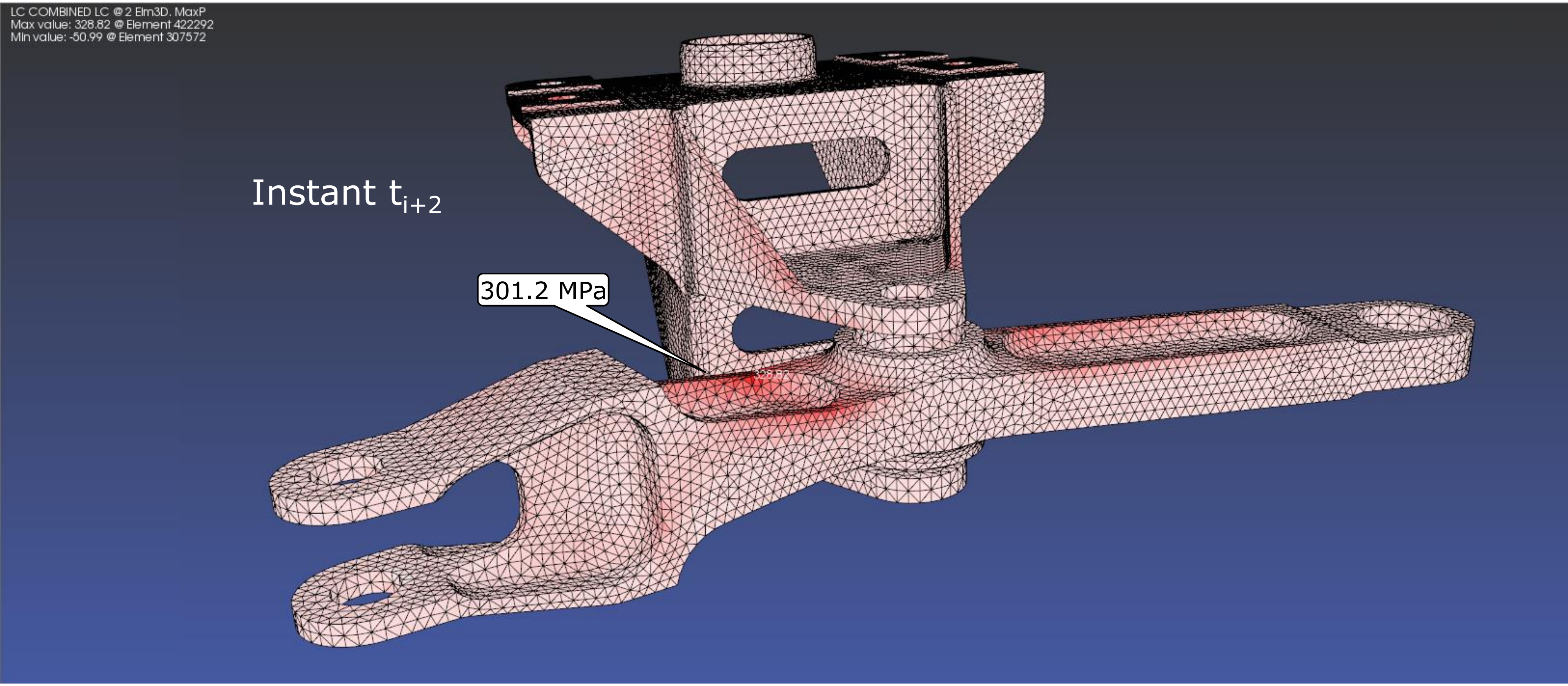
MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

- Multiaxial Proportional Loading Conditions (software LIFING)



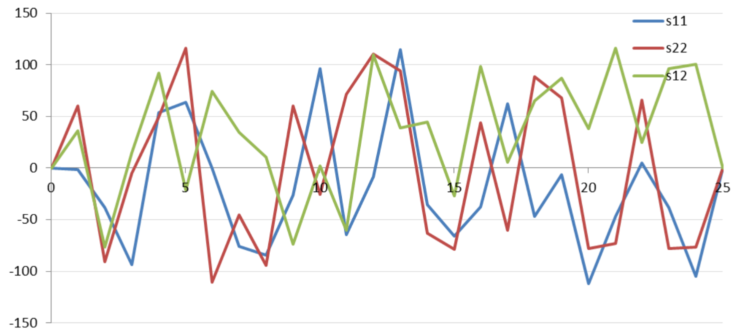
MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

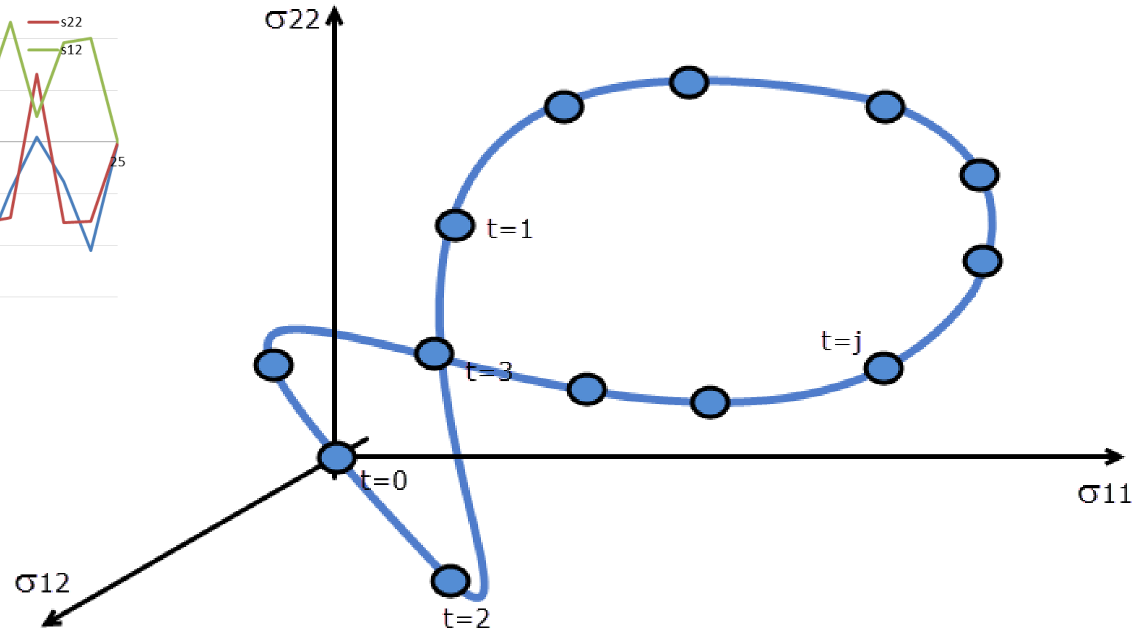
- Multiaxial Proportional Loading Conditions (software LIFING)



MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

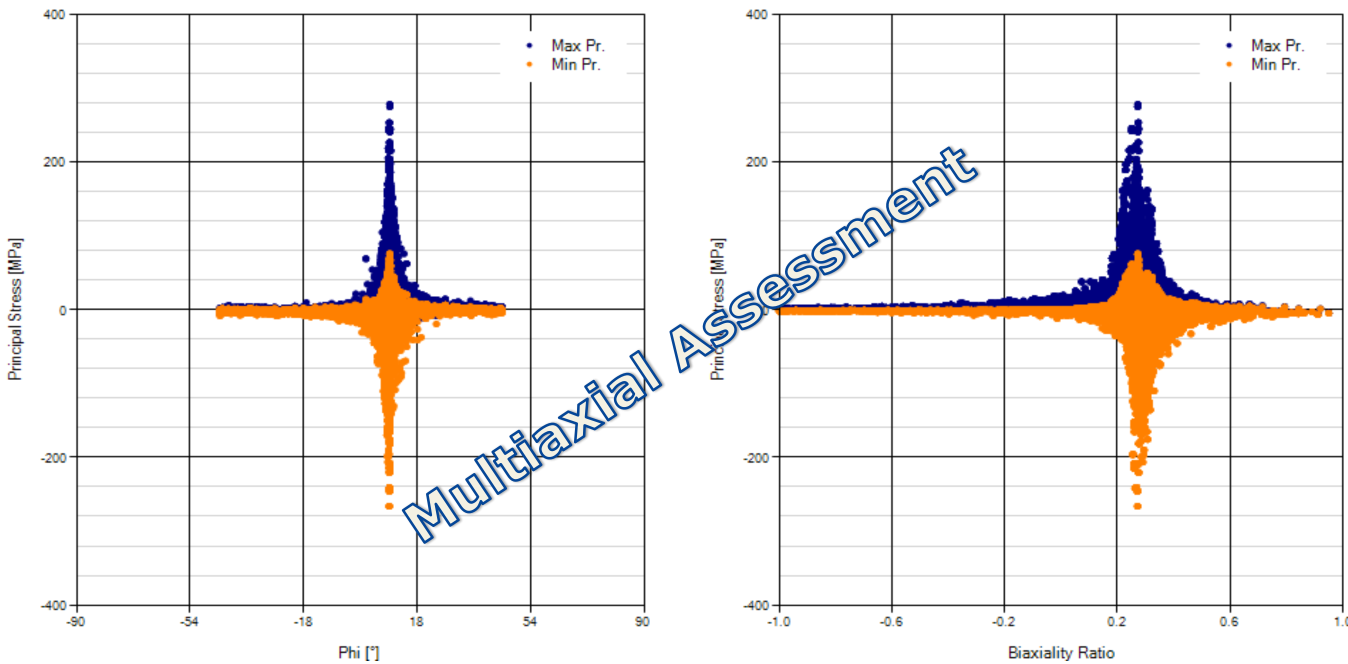
- Multiaxial – Non-Proportional Loadings.
  - This situation is the general one, when the structure is subjected to multiple loads which vary in time not in phase.





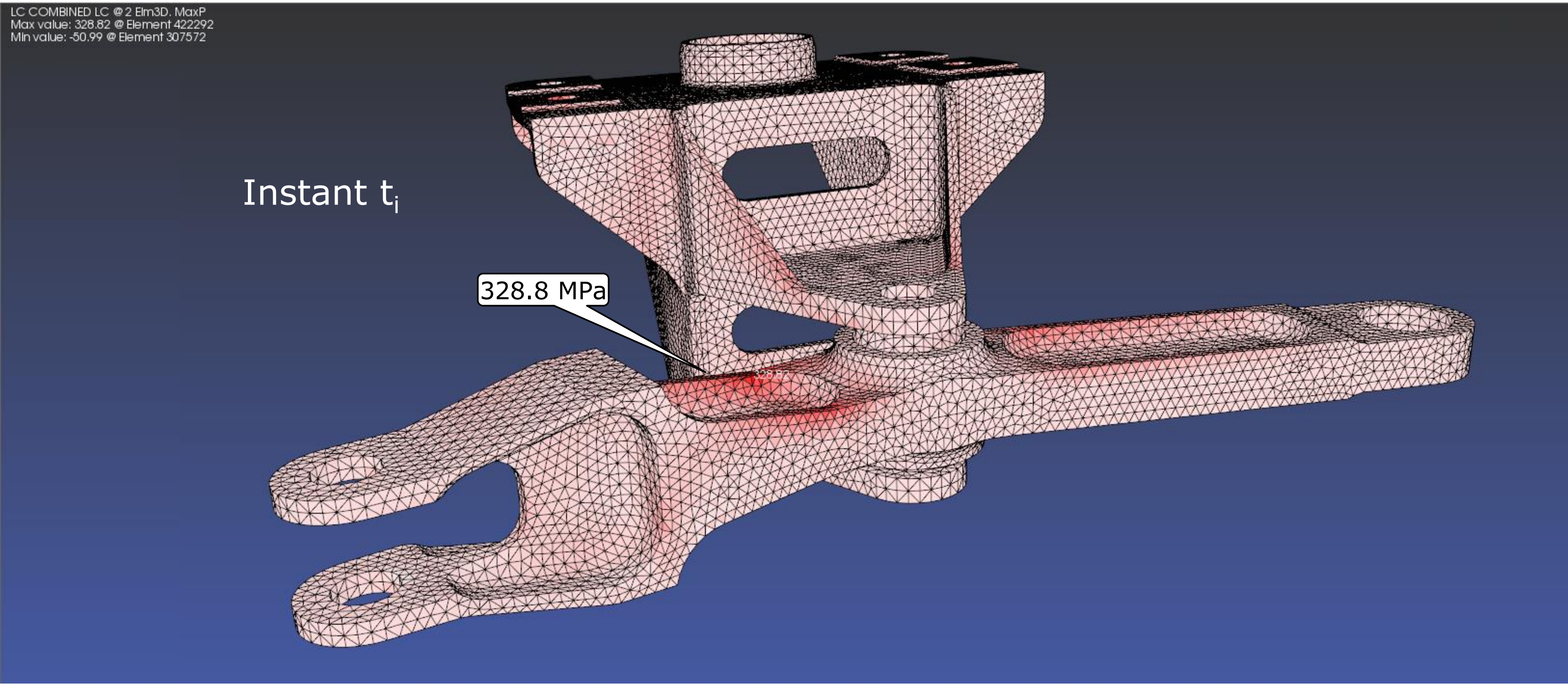
MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

- Multiaxial – Non-Proportional Loadings.
  - If stress components are plotted in a chart, the points do not lay on a not straight line.
  - Similarly, stress principal directions and bi-axiality ratios change over time.



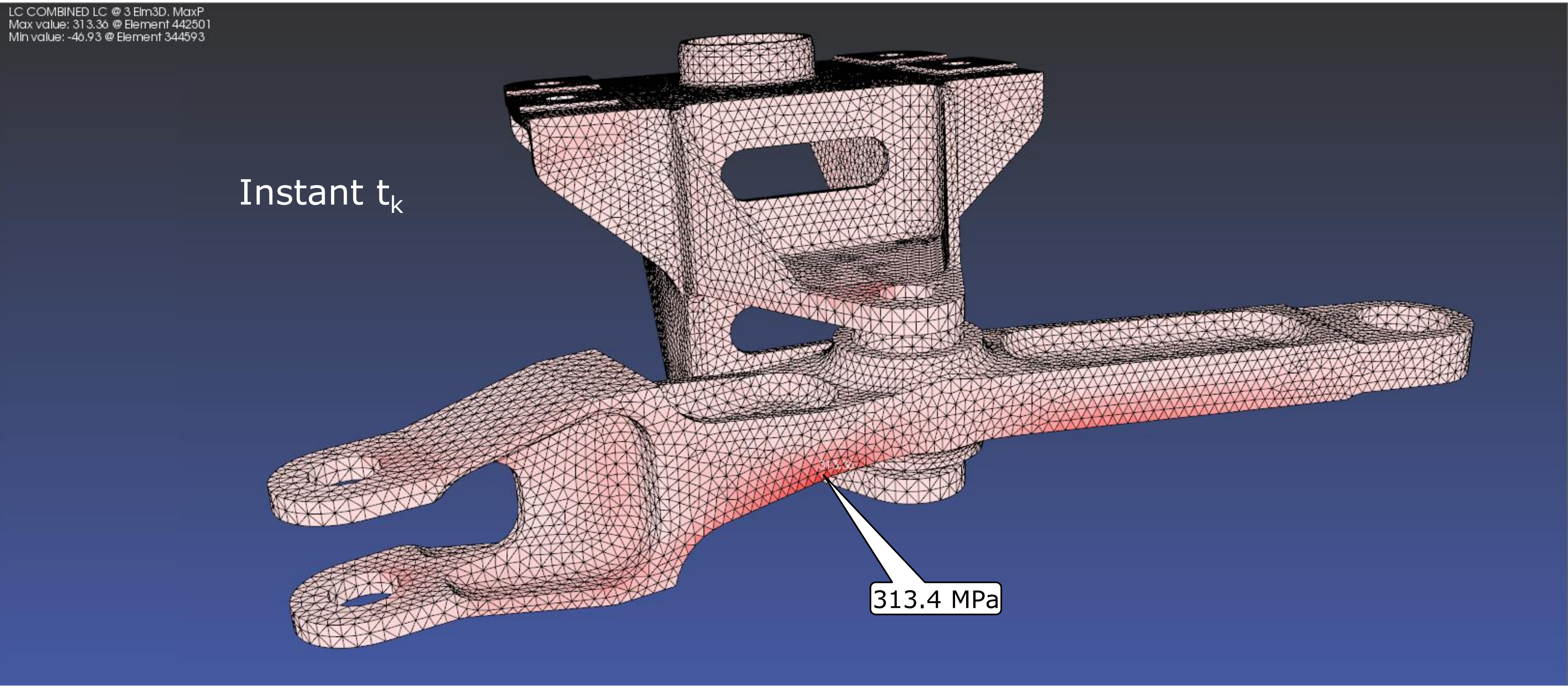
MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

- Multiaxial Non-Proportional Loading Conditions (software LIFING)



MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

- Multiaxial Non-Proportional Loading Conditions (software LIFING)



MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

- Biaxiality Ratio is usually defined as the ratio between the Minimum and Maximum (in magnitude) Principal Stress.

$$[\sigma] = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \qquad \lambda = \frac{\sigma_2}{\sigma_1}$$

The principal stress σ_1 is the one characterized by highest magnitude

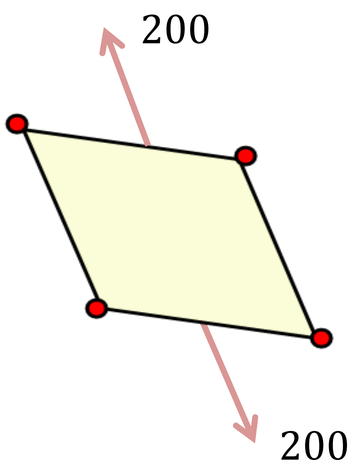
- By definition, the bi-axiality ratio λ spans between -1 and 1.
- In case of uniaxial stress tensor, the bi-axiality λ ratio is zero.
- What could be the implication of $\lambda \neq 0$ in a calculation which neglects the presence of a second stress? (*)

(*) This is the case, for example, when the analysis is carried out just looking at the Maximum Principal stress time history

MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

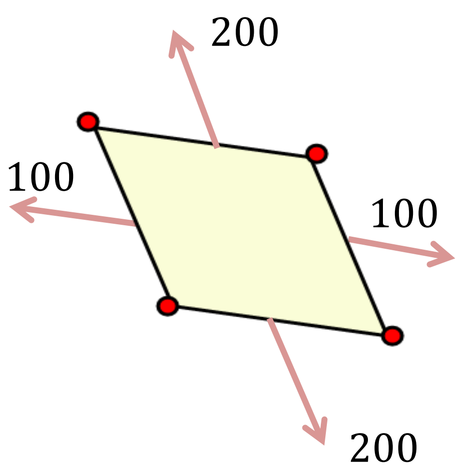
- What would happen if we run a fatigue analysis at three different notches loaded with the same time history of loads (Multiaxial Proportional Loading), where the following three reference stress tensors are given?

Case 1



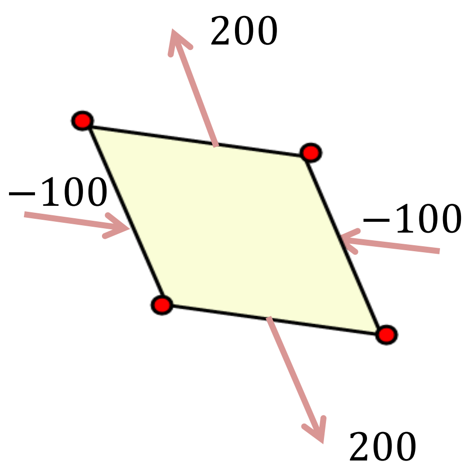
$$\begin{bmatrix} 200 & 0 \\ 0 & 0 \end{bmatrix}$$

Case 2



$$\begin{bmatrix} 200 & 0 \\ 0 & 100 \end{bmatrix}$$

Case 3



$$\begin{bmatrix} 200 & 0 \\ 0 & -100 \end{bmatrix}$$

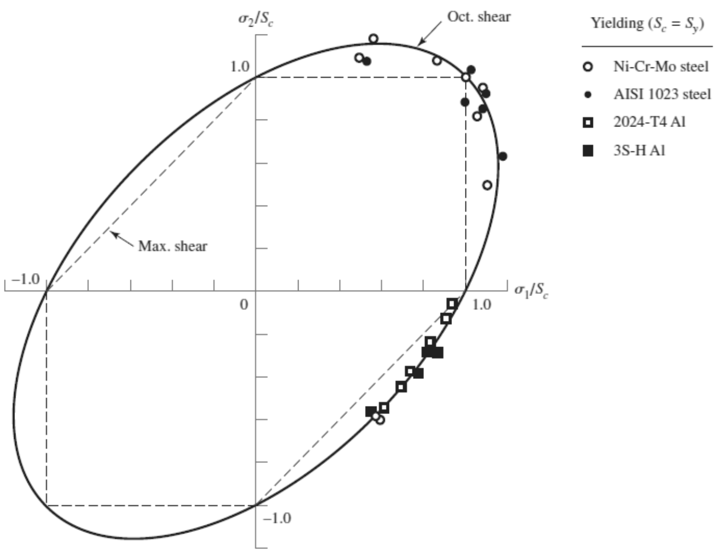
MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

- The fatigue analysis performed on the basis of the Max Principal Stress would deliver the same result for all the three cases. However if we calculate the Von Mises stresses we have:

- Case 1. $\mathrm{VM} = \sqrt{{\sigma_1}^2 + {\sigma_2}^2 - \sigma_1 \cdot \sigma_2} = 200\mathrm{MPa}$

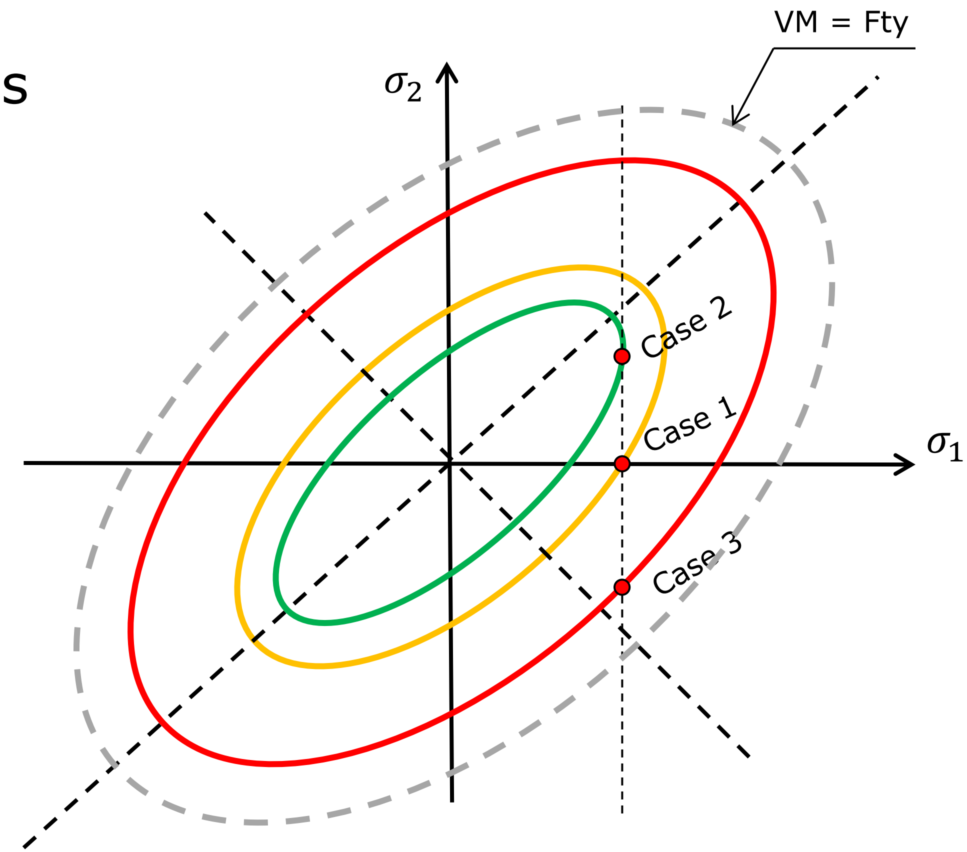
- Case 2. $\mathrm{VM} = \sqrt{{\sigma_1}^2 + {\sigma_2}^2 - \sigma_1 \cdot \sigma_2} = 173.2\mathrm{MPa}$

- Case 3. $\mathrm{VM} = \sqrt{{\sigma_1}^2 + {\sigma_2}^2 - \sigma_1 \cdot \sigma_2} = 264.6\mathrm{MPa}$



MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

- Yield surfaces



Case 1: 200 MPa

Case 2: 173.2 MPa

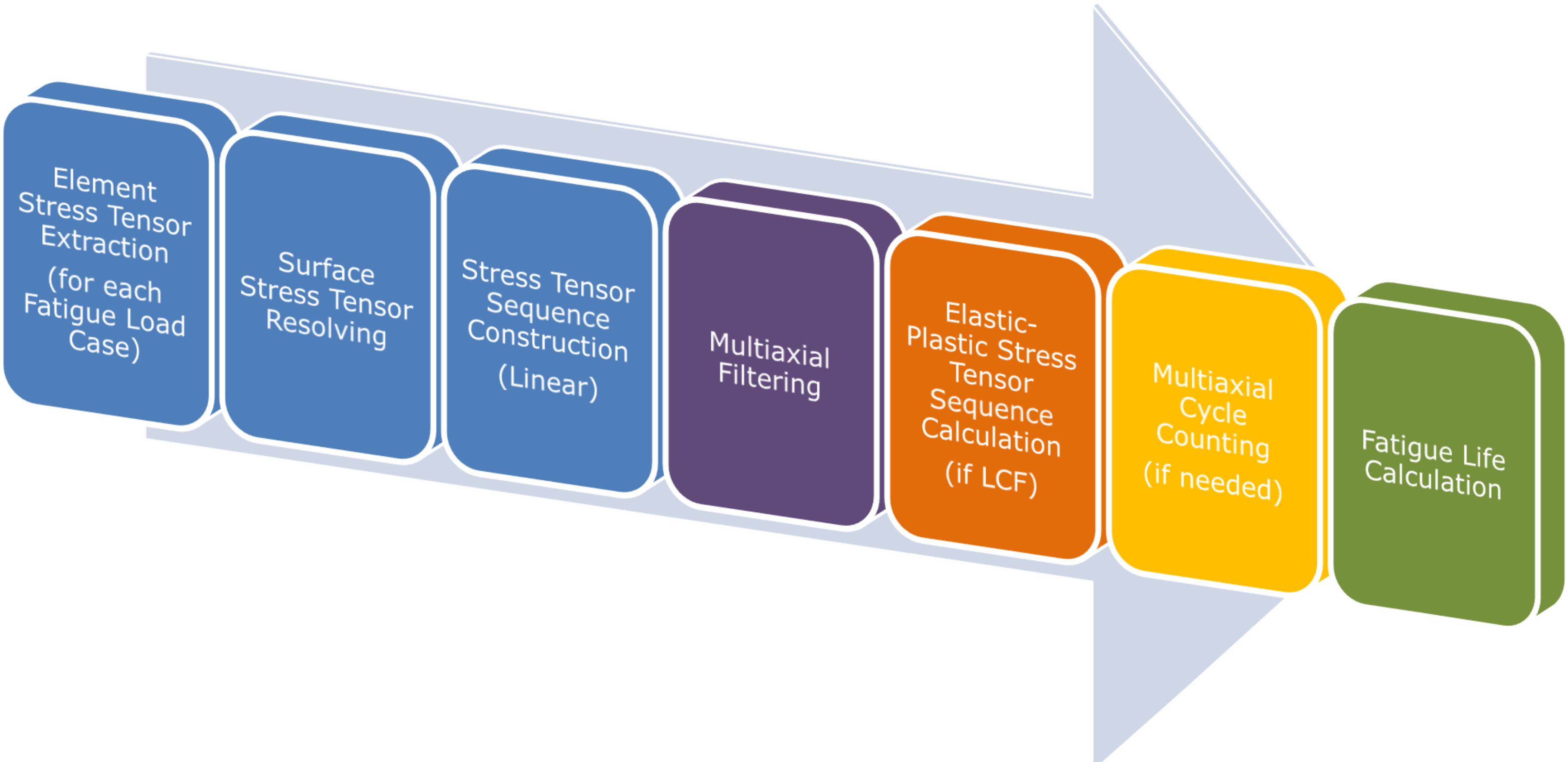
Case 3: 264.6 MPa

- Can the three cases be equally damaging?

MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

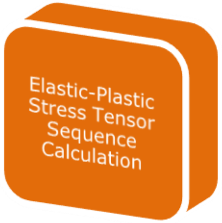
- The fatigue (crack initiation) analysis performed on the basis of the Max Principal Stress, inherits two fundamental errors:
  - (1) Assume the problem is Multiaxial Proportional Loading.
    - The impact of biaxiality ratio is ignored, meaning that:
      - CONSERVATIVE ERROR if $\lambda>0$
      - UNCONSERVATIVE ERROR if $\lambda<0$
  - (2) Assume the problem is Multiaxial Non-Proportional Loading.
    - The Maximum Principal Plane ROTATES over the time (and biaxiality ratio changes as well), meaning that:
      - At each instant in the time history the structure wants to crack at different planes, whereas the Maximum Principal is an ‘invariant’ (i.e. plane insensitive)

MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS



MULTIAXIAL FEM ASSISTED FATIGUE ANALYSIS

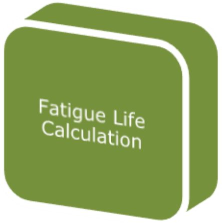
- The Non-Proportionality significantly increases the problem complexity, because of the following issues:



– Solving Cyclic Plasticity (in case of LCF) with multiple stress components is way more complex (many methods are available), therefore the calculation of elastic-plastic stress-strains out of elastic FEM calculated stress-strains is a very complex issue.

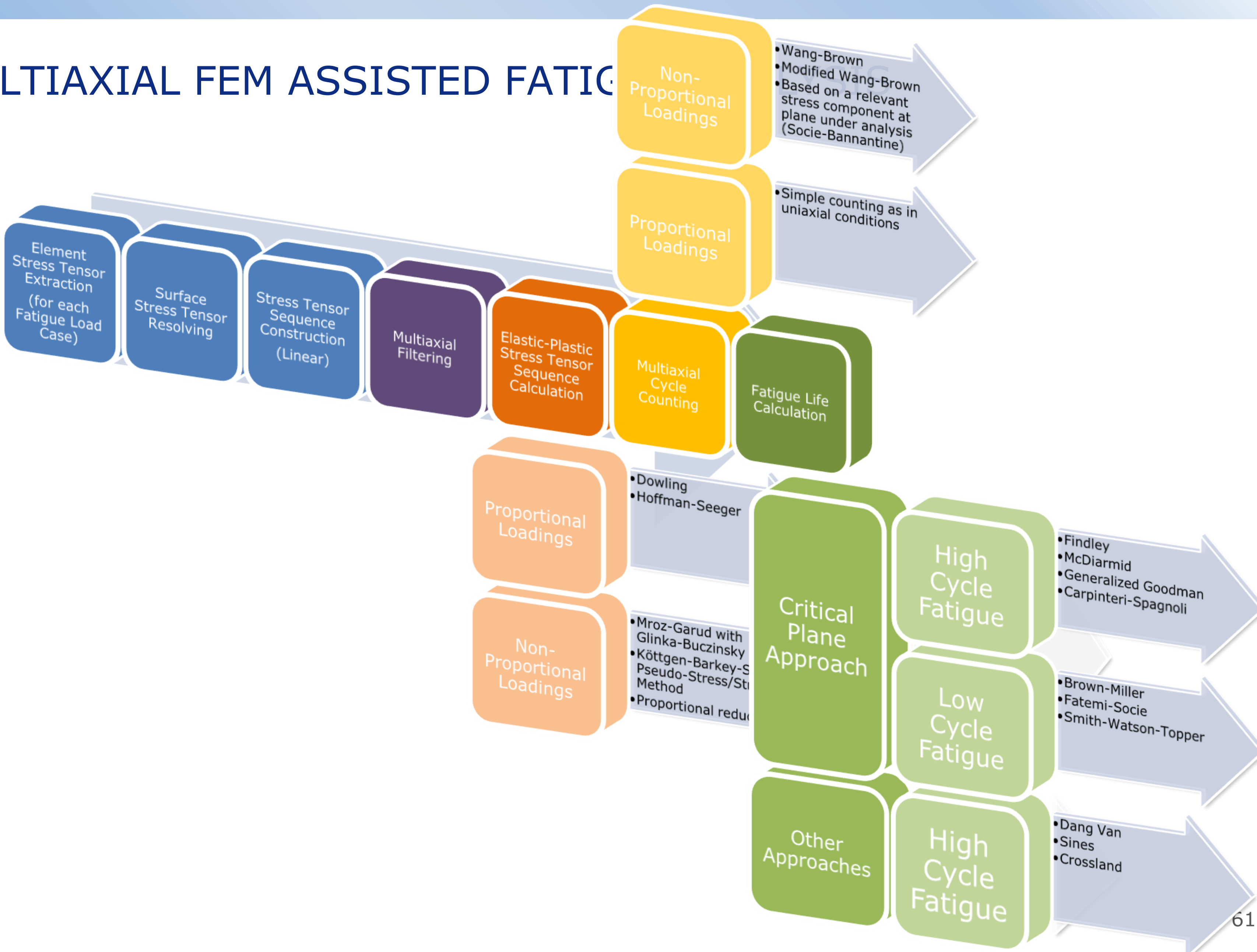
  (Simple approaches are available in case of Proportional Loadings)

– If we were able to calculate elastic-plastic stress-strains, defining cycles within a Stress Tensor time history where components change over the time not in phase is complex (Wang-Brown method is proposed in literature, however some analysis methods do not require special sequence counting).



– What is the best fatigue parameter, combination of stress components, if multiple stress components vary over the time?

MULTIAXIAL FEM ASSISTED FATIG



DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Dowling and Hoffman-Seeger devised two similar methods to calculate elastic-plastic stress status for the Multiaxial Proportional Loadings problems.

- Both are based on the modification of the Strain based method described in the previous chapters, in order to take into account the bi-axiality ratio.

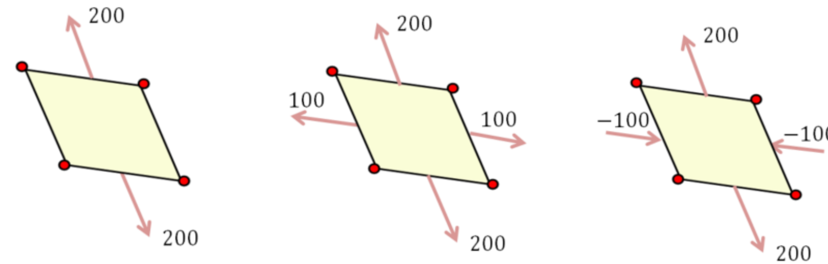
- It is demonstrated, with the application of such methods, that the presence of other additional has an impact on the fatigue results.

DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Dowling method
  - This method assumes that both principal strain ratio and principal stress ratio (both called bi-axiality ratio) are constant

$$\varphi = {}^{\varepsilon_2}/_{\varepsilon_1} = {}^{\varepsilon_2^e}/_{\varepsilon_1^e}$$

$$\lambda = {}^{\sigma_2}/_{\sigma_1} = {}^{\sigma_2^e}/_{\sigma_1^e}$$



$\lambda, \varphi = 0$ $\quad$ $\lambda, \varphi > 0$ $\quad$ $\lambda, \varphi < 0$

  - This is an approximation, because when loading in plastic range the two ratios are not constant. The implied error is proportional to the amount of plastic strain, therefore the error is small for small plastic strains.

DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Dowling method
  - The superscript "e" denotes elastic values, the subscripts 1 and 2 denote respectively Max Principal and Min Principal values:

$$\sigma_1 = \frac{\sigma_{11}+\sigma_{22}}{2} + \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2} \qquad \sigma_2 = \frac{\sigma_{11}+\sigma_{22}}{2} - \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$$

$$\sigma_{eq} = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} = \sigma_1\sqrt{\lambda^2 - \lambda + 1}$$

The Von Mises stress is considered as equivalent stress

$$\varepsilon_1 = \frac{\varepsilon_{11}+\varepsilon_{22}}{2} + \sqrt{\left(\frac{\varepsilon_{11}-\varepsilon_{22}}{2}\right)^2 - \varepsilon_{12}^2} = \frac{\varepsilon_{11}+\varepsilon_{22}}{2} + \sqrt{\left(\frac{\varepsilon_{11}-\varepsilon_{22}}{2}\right)^2 - \left(\frac{\gamma_{12}}{2}\right)^2}$$

$$\varepsilon_2 = \frac{\varepsilon_{11}+\varepsilon_{22}}{2} - \sqrt{\left(\frac{\varepsilon_{11}-\varepsilon_{22}}{2}\right)^2 - \varepsilon_{12}^2} = \frac{\varepsilon_{11}+\varepsilon_{22}}{2} - \sqrt{\left(\frac{\varepsilon_{11}-\varepsilon_{22}}{2}\right)^2 - \left(\frac{\gamma_{12}}{2}\right)^2}$$

DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Dowling method
  - From the constitutive equations:

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) = \frac{1}{E}(1-\nu\lambda)\sigma_1 = \frac{\sigma_1}{E^*} \qquad \text{with} \qquad E^* = \frac{E}{1-\nu\lambda}$$

$$\varepsilon_2 = \varphi\varepsilon_1 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) = \frac{1}{E}(\lambda - \nu)\sigma_1$$

$$\varphi\varepsilon_1 = \varphi\frac{1}{E}(1-\nu\lambda)\sigma_1 = \frac{1}{E}(\lambda-\nu)\sigma_1 \quad \rightarrow \quad (\lambda-\nu) = \varphi(1-\nu\lambda)$$

  - The maximum strain is rearranged as follows:

$$\varepsilon_1 = {\varepsilon_1}^e + {\varepsilon_1}^p = \frac{1}{E}(1-\nu\lambda)\sigma_1 + \frac{1}{E_p}\left(1-\nu_p\lambda\right)\sigma_1 = \frac{\sigma_1}{E^*} + \frac{\sigma_1}{E_p}(1-0.5\lambda) =$$

$$= \frac{\sigma_1}{E^*} + \frac{\sigma_1}{\sigma_{eq}}{\varepsilon_{eq}}^p(1-0.5\lambda)$$

(being in full plasticity $\nu_p = 0.5$)

DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Dowling method

$$\varepsilon_{eq}{}^{p} = \left(\frac{\sigma_{eq}}{K'}\right)^{\frac{1}{n\prime}} \quad \rightarrow \quad \frac{\varepsilon_{eq}{}^{p}}{\sigma_{eq}} = \frac{1}{\sigma_{eq}}\left(\frac{\sigma_{eq}}{K'}\right)^{\frac{1}{n\prime}} = \frac{(\lambda^2 - \lambda + 1)^{-\frac{1}{2}+\frac{1}{2n\prime}}}{\sigma_1}\left(\frac{\sigma_1}{K'}\right)^{\frac{1}{n\prime}}$$

$$\varepsilon_1 = \frac{\sigma_1}{E^*} + (\lambda^2 - \lambda + 1)^{\frac{n'-1}{2n\prime}}(1 - 0.5\lambda)\left(\frac{\sigma_1}{K'}\right)^{\frac{1}{n\prime}} = \frac{\sigma_1}{E^*} + \left(\frac{\sigma_1}{K'^*}\right)^{\frac{1}{n\prime}}$$

$$K'^* = \frac{K'}{(1 - 0.5\lambda)^{n\prime}}(\lambda^2 - \lambda + 1)^{\frac{n'-1}{2}}$$

   - Dowling method uses the same Strain based approach defined in previous chapter, but the cyclic Ramberg-Osgood equation coefficient E and K' are replaced by E^* *and* K'^*

DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Dowling method
  - The problem is solved as in the uniaxial approach, using the Ramberg-Osgood equation in conjunction with the Neuber (or Glinka E.S.E.D.) equation

$$\left.\begin{aligned} \varepsilon_1 &= \frac{\sigma_1}{E^*} + \left(\frac{\sigma_1}{K'^*}\right)^{\frac{1}{n'}} \\ {\sigma_{eq}}^e {\varepsilon_{eq}}^e &= \frac{\left({\sigma_{eq}}^e\right)^2}{E} = \frac{{\sigma_1}^2}{E^*} + \bar{\alpha}_U \sigma_1 \left(\frac{\sigma_1}{K'^*}\right)^{\frac{1}{n'}} \end{aligned}\right\} \text{Monotonic}$$

$$\left.\begin{aligned} \Delta\varepsilon_1 &= \frac{\Delta\sigma_1}{E^*} + \left(\frac{\Delta\sigma_1}{2K'^*}\right)^{\frac{1}{n'}} \\ \frac{\left(\Delta{\sigma_{eq}}^e\right)^2}{E} &= \frac{\Delta{\sigma_1}^2}{E^*} + 2\bar{\alpha}_U \Delta\sigma_1 \left(\frac{\Delta\sigma_1}{2K'^*}\right)^{\frac{1}{n'}} \end{aligned}\right\} \text{Cyclic}$$

DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Dowling method
  - Solving the two equations, numerically, σ_1 and ε_1 are calculated. The other stresses and strains are then derived:

$$\sigma_2 = \lambda \cdot \sigma_1 \qquad \varepsilon_2 = \varphi \cdot \varepsilon_1 = \frac{\lambda - \nu}{1 - \lambda\nu}\varepsilon_1$$

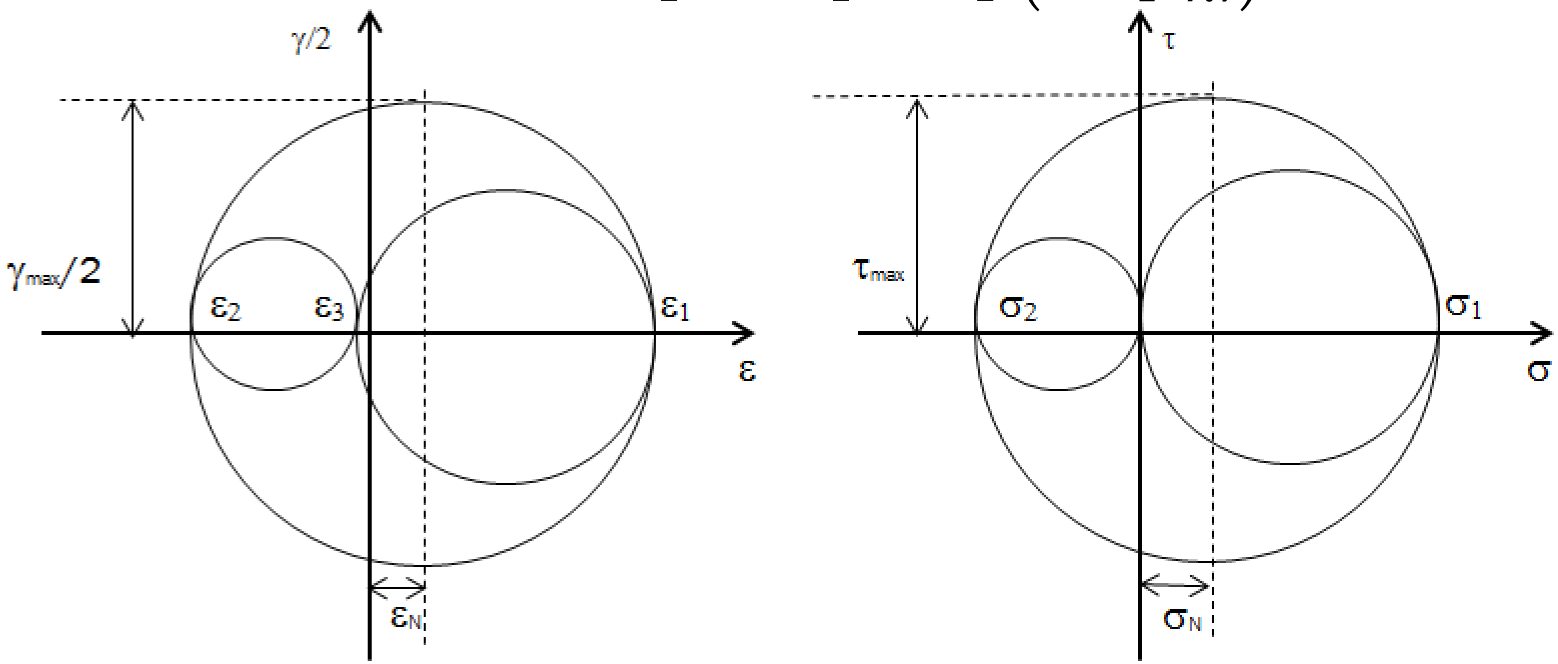
$$\varepsilon_3 = -\frac{\nu'}{E}(\sigma_1 + \sigma_2) = -\frac{\nu'}{E}(1+\lambda)\sigma_1 = -\frac{\nu'}{E}(1+\lambda)\frac{\varepsilon_1}{\frac{1}{E}(1-\nu'\lambda)} = -\nu'\frac{1+\lambda}{1-\nu'\lambda}\varepsilon_1$$

  - being ν' the Poisson ratio in plastic region, calculated as

$$\nu' = \nu_p - \left(\nu_p - \nu\right)\left|\frac{\sigma_1}{E^* \cdot \varepsilon_1}\right| = 0.5 - (0.5 - \nu)\left|\frac{\sigma_1}{E^* \cdot \varepsilon_1}\right|$$

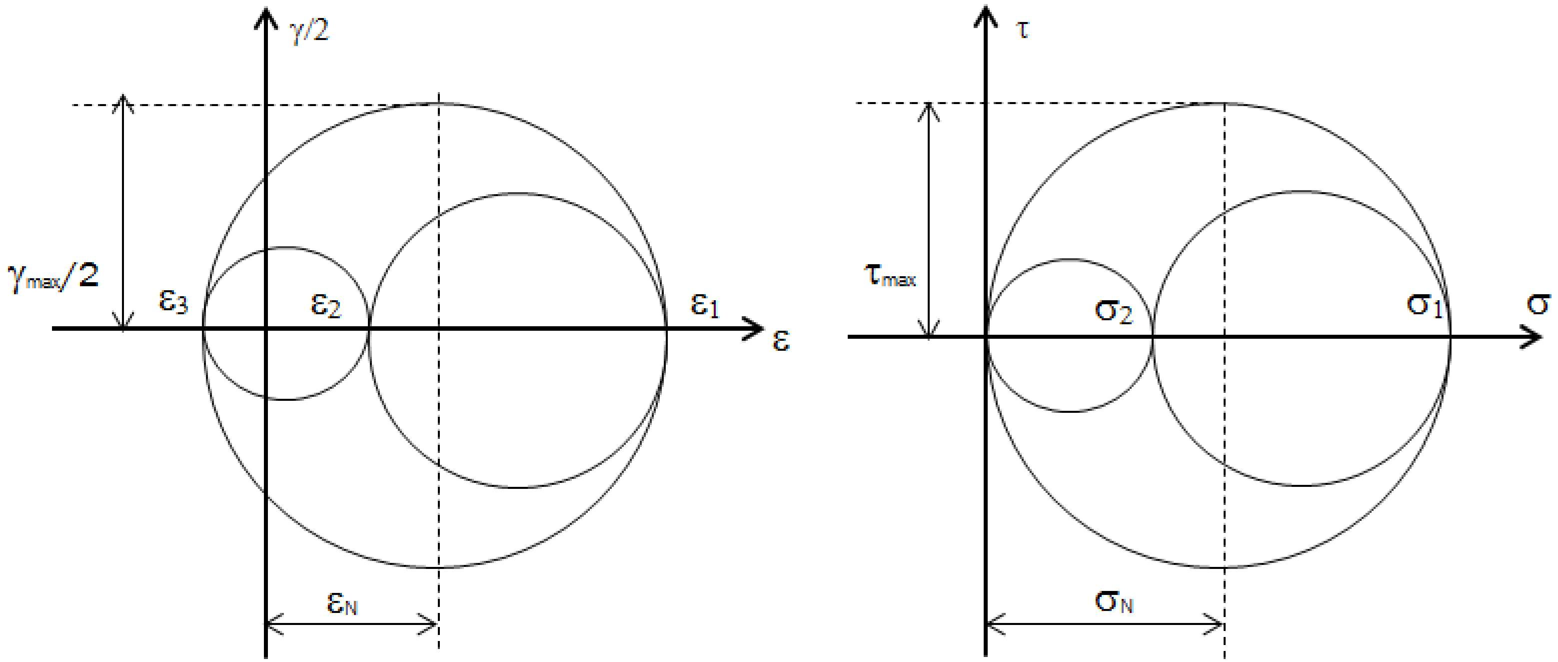
DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Dowling method
  - Depending on the bi-axiality ratio λ the calculation of the shear strain changes.
  - In case of $\lambda \leq 0$: $$\frac{\gamma_{max}}{2} = \frac{\varepsilon_1 - \varepsilon_2}{2} = \frac{\varepsilon_1}{2}\left(1 - \frac{\lambda - \nu}{1 - \lambda\nu}\right)$$



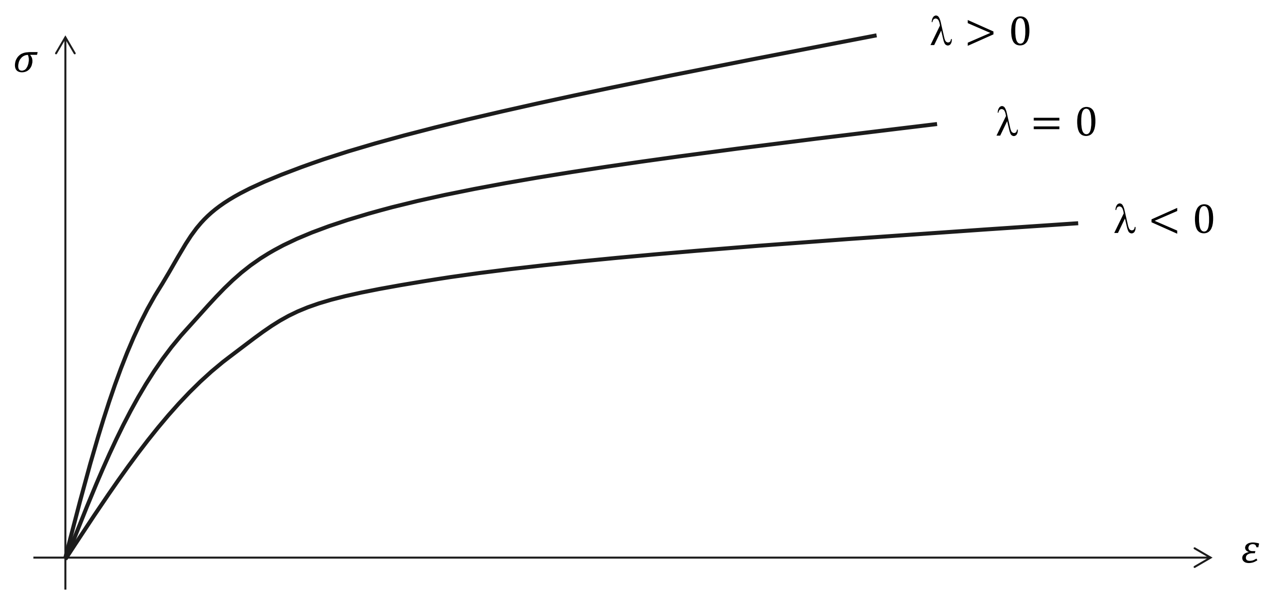
DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Dowling method
  - In case of $\lambda > 0$: $$\frac{\gamma_{max}}{2} = \frac{\varepsilon_1 - \varepsilon_3}{2} = \frac{\varepsilon_1}{2}\left(1 + \nu' \frac{1+\lambda}{1-\lambda\nu'}\right)$$



DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Dowling method
  - With respect to a uniaxial fatigue analysis, the presence of a bi-axiality ratio leads to results that would be conservative or unconservative if the bi-axiality ratio would be ignored:
    - $\lambda < 0 \Rightarrow$ UNCONSERVATIVE
    - $\lambda > 0 \Rightarrow$ CONSERVATIVE



DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Hoffman-Seeger method
  - This method assumes that only the strain ratio φ is constant (not also the stress ratio λ), which is an assumption more adherent to the experimental evidence
  - The approach considers same equations as for the uniaxial case, applied to an equivalent stress (signed Von Mises)

$$\sigma_{eq} = \frac{\sigma_1}{|\sigma_1|}\sqrt{{\sigma_1}^2 + {\sigma_2}^2 - \sigma_1 \cdot \sigma_2}$$

  - The Ramberg-Osgood equation gets the form

$$\varepsilon_{eq} = \frac{\sigma_{eq}}{E} + \left(\frac{\sigma_{eq}}{K'}\right)^{\frac{1}{n\prime}}$$

DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Hoffman-Seeger method

$$\sigma_{eq}{}^{e}\varepsilon_{eq}{}^{e} = \frac{\left(\sigma_{eq}{}^{e}\right)^{2}}{E} = \frac{\sigma_{eq}{}^{2}}{E} + \bar{\alpha}_{U}\sigma_{eq}\left(\frac{\sigma_{eq}}{K'}\right)^{\frac{1}{n'}}$$

Monotonic

$$\frac{\left(\Delta\sigma_{eq}{}^{e}\right)^{2}}{E} = \frac{\Delta\sigma_{eq}{}^{2}}{E} + 2\bar{\alpha}_{U}\Delta\sigma_{eq}\left(\frac{\Delta\sigma_{eq}}{2K'}\right)^{\frac{1}{n'}}$$

Cyclic

  - By solving the Ramberg-Osgood and one of the two above, the equivalent stress and strains are obtained, σ_{eq} and ε_{eq}

DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Hoffman-Seeger method
  - The Poisson ratio in plastic region is

$$\nu' = \nu_p - \left(\nu_p - \nu\right)\left|\frac{\sigma_{eq}}{E \cdot \varepsilon_{eq}}\right| = 0.5 - (0.5 - \nu)\left|\frac{\sigma_{eq}}{E \cdot \varepsilon_{eq}}\right|$$

  - And the bi-axiality ratio in plastic domain is calculated (from the constitutive equations)

$$\lambda = \frac{\sigma_2}{\sigma_1} = \frac{\varphi + \nu'}{1 + \nu'\varphi}$$

  - With Dowling approach this was assumed constant; with Hoffmann-Seeger it is not constant.

DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Hoffman-Seeger method
  - The max principal stresses are

$$\sigma_1 = \sigma_{eq} \frac{1}{\sqrt{1 - \lambda + \lambda^2}} \qquad \sigma_2 = \lambda \cdot \sigma_1 = \sigma_{eq} \frac{\lambda}{\sqrt{1 - \lambda + \lambda^2}}$$

  - The strain components, from the constitutive equations, are

$$\varepsilon_1 = \frac{\varepsilon_{eq}}{\sigma_{eq}} (\sigma_1 - \nu' \sigma_2) = \varepsilon_{eq} \frac{1 - \nu' \lambda}{\sqrt{1 - \lambda + \lambda^2}}$$

$$\varepsilon_2 = \frac{\varepsilon_{eq}}{\sigma_{eq}} (\sigma_2 - \nu' \sigma_1) = \varepsilon_{eq} \frac{\lambda - \nu'}{\sqrt{1 - \lambda + \lambda^2}}$$

$$\varepsilon_3 = \frac{\varepsilon_{eq}}{\sigma_{eq}} [-\nu' (\sigma_1 + \sigma_2)] = -\nu' \cdot \varepsilon_{eq} \frac{1 + \lambda}{\sqrt{1 - \lambda + \lambda^2}}$$

DOWLING AND HOFFMAN-SEEGER METHODS FOR MULTIAXIAL PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Hoffman-Seeger method
  - Depending on the bi-axiality ratio λ the calculation of the shear strain changes.
  - In case of $\lambda \leq 0$:

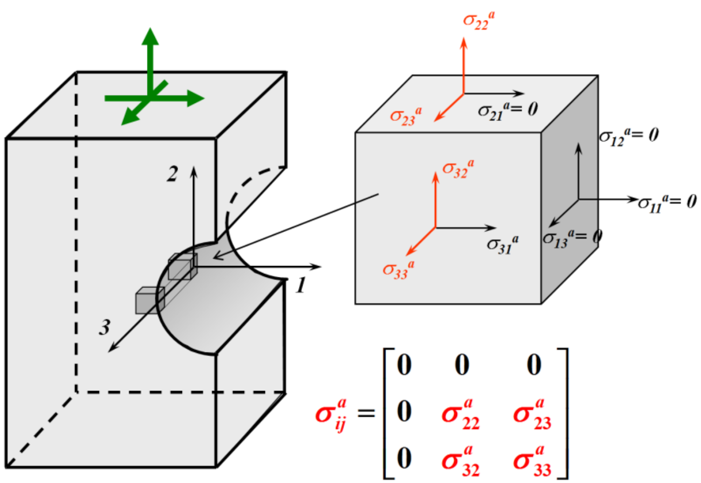
$$\frac{\gamma_{max}}{2} = \frac{\varepsilon_1 - \varepsilon_2}{2} = \frac{\varepsilon_{eq}}{2} \frac{1 - \nu'\lambda - \lambda + \nu'}{\sqrt{1 - \lambda + \lambda^2}}$$

  - In case of $\lambda > 0$:

$$\frac{\gamma_{max}}{2} = \frac{\varepsilon_1 - \varepsilon_3}{2} = \frac{\varepsilon_{eq}}{2} \frac{1 - \nu' - 2\nu'\lambda}{\sqrt{1 - \lambda + \lambda^2}}$$

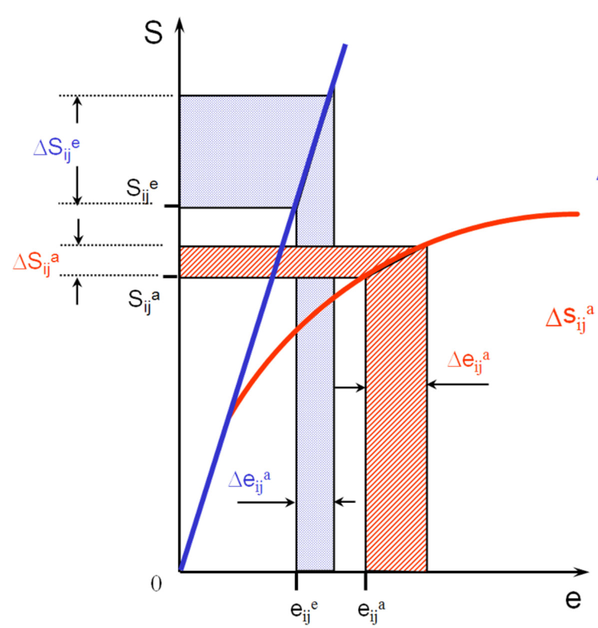
GLINKA-BUCZYNSKY METHOD FOR MULTIAXIAL NON-PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Many solution schemes for generic Non-Proportional Loading problem have been developed. Some, like the one from **Glinka-Buczynsky**, are computationally demanding as they involve the incremental solution of a large set of non-linear equations

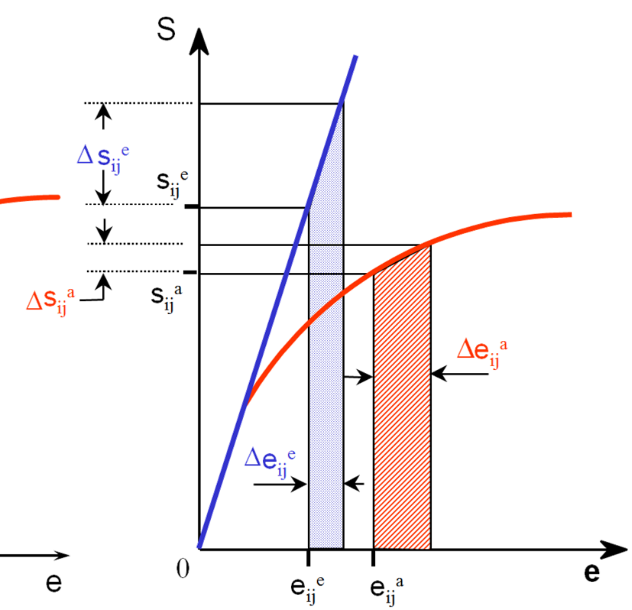


$$\sigma_{ij}^{a} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22}^{a} & \sigma_{23}^{a} \\ 0 & \sigma_{32}^{a} & \sigma_{33}^{a} \end{bmatrix}$$

Incremental Neuber's Rule



Incremental ESED Method



In the deviatoric space: $S^{e}_{\alpha\beta}\Delta e^{e}_{\alpha\beta} + e^{e}_{\alpha\beta}\Delta S^{e}_{\alpha\beta} = S^{a}_{\alpha\beta}\Delta e^{a}_{\alpha\beta} + e^{a}_{\alpha\beta}\Delta S^{a}_{\alpha\beta}$ $\quad S^{e}_{\alpha\beta}\Delta e^{e}_{\alpha\beta} = S^{a}_{\alpha\beta}\Delta e^{a}_{\alpha\beta}$

GLINKA-BUCZYNSKY METHOD FOR MULTIAXIAL NON-PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

Incremental Multiaxial Neuber's Rule

$$S_{22}^{e}\Delta e_{22}^{e}+e_{22}^{e}\Delta S_{22}^{e}=S_{22}^{a}\Delta e_{22}^{a}+e_{22}^{a}\Delta S_{22}^{a}$$

$$S_{33}^{e}\Delta e_{33}^{e}+e_{33}^{e}\Delta S_{33}^{e}=S_{33}^{a}\Delta e_{33}^{a}+e_{33}^{a}\Delta S_{33}^{a}$$

$$S_{23}^{e}\Delta e_{23}^{e}+e_{23}^{e}\Delta S_{23}^{e}=S_{23}^{a}\Delta e_{23}^{a}+e_{23}^{a}\Delta S_{23}^{a}$$

$$\Delta e_{22}^{a}=\frac{\Delta S_{22}^{a}}{2G}+\frac{3}{2}\frac{\Delta\varepsilon_{eq}^{a}}{\sigma_{eq}^{a}}S_{22}^{a}$$

$$\Delta e_{33}^{a}=\frac{\Delta S_{33}^{a}}{2G}+\frac{3}{2}\frac{\Delta\varepsilon_{eq}^{a}}{\sigma_{eq}^{a}}S_{33}^{a}$$

$$\Delta e_{23}^{a}=\frac{\Delta S_{23}^{a}}{2G}+\frac{3}{2}\frac{\Delta\varepsilon_{eq}^{a}}{\sigma_{eq}^{a}}S_{23}^{a}$$

$$\Delta e_{11}^{a}=-\frac{\Delta S_{22}^{a}+\Delta S_{22}^{a}}{2G}-\frac{3}{2}\frac{\Delta\varepsilon_{eq}^{a}}{\sigma_{eq}^{a}}\left(S_{22}^{a}+S_{33}^{a}\right)$$

Incremental Multiaxial ESED Rule

$$S_{22}^{e}\Delta e_{22}^{e}=S_{22}^{a}\Delta e_{22}^{a}$$

$$S_{33}^{e}\Delta e_{33}^{e}=S_{33}^{a}\Delta e_{33}^{a}$$

$$S_{23}^{e}\Delta e_{23}^{e}=S_{23}^{a}\Delta e_{23}^{a}$$

$$\Delta e_{22}^{a}=\frac{\Delta S_{22}^{a}}{2G}+\frac{3}{2}\frac{\Delta\varepsilon_{eq}^{a}}{\sigma_{eq}^{a}}S_{22}^{a}$$

$$\Delta e_{33}^{a}=\frac{\Delta S_{33}^{a}}{2G}+\frac{3}{2}\frac{\Delta\varepsilon_{eq}^{a}}{\sigma_{eq}^{a}}S_{33}^{a}$$

$$\Delta e_{23}^{a}=\frac{\Delta S_{23}^{a}}{2G}+\frac{3}{2}\frac{\Delta\varepsilon_{eq}^{a}}{\sigma_{eq}^{a}}S_{23}^{a}$$

$$\Delta e_{11}^{a}=-\frac{\Delta S_{22}^{a}+\Delta S_{22}^{a}}{2G}-\frac{3}{2}\frac{\Delta\varepsilon_{eq}^{a}}{\sigma_{eq}^{a}}\left(S_{22}^{a}+S_{33}^{a}\right)$$

GLINKA-BUCZYNSKY METHOD FOR MULTIAXIAL NON-PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Blue terms: Elastic input (from FEM linear elastic analysis)
- Red terms: Elastic-plastic output
- 7 equations in 8 unknowns (Δe^a_{11}, Δe^a_{22}, Δe^a_{33}, Δe^a_{23}, ΔS^a_{22}, ΔS^a_{33}, ΔS^a_{23}, $\Delta \varepsilon^a_{eq}$) where the term

$$\Delta\varepsilon_{eq}{}^{a} = \frac{df\left(\sigma_{eq}{}^{a}\right)}{d\sigma_{eq}{}^{a}}\Delta\varepsilon_{eq}{}^{a} \quad \rightarrow \quad \frac{3}{2}\frac{\Delta\varepsilon_{eq}{}^{a}}{\sigma_{eq}{}^{a}} = \frac{3}{2}\frac{\Delta\sigma_{eq}{}^{a}}{C\,\sigma_{eq}{}^{a}}$$

C = Plastic modulus

- is calculated with the *Mróz-Garud* Multi-Surface Model (later described), representing the 8th condition for the solution
- At each increment the terms e^a_{11}, e^a_{22}, e^a_{33}, e^a_{23}, S^a_{22}, S^a_{33}, S^a_{23} are derived from the calculated increments and previous iteration condition

GLINKA-BUCZYNSKY METHOD FOR MULTIAXIAL NON-PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- From deviatoric quantities, the stresses and strains are derived

$$\begin{cases} S_{22}^{a} = \sigma_{22}^{a} - \frac{1}{3}\left(\sigma_{22}^{a} + \sigma_{33}^{a}\right) \\ S_{33}^{a} = \sigma_{33}^{a} - \frac{1}{3}\left(\sigma_{22}^{a} + \sigma_{33}^{a}\right) \\ S_{23}^{a} = \sigma_{23}^{a} \end{cases} \longrightarrow \sigma_{22}^{a},\ \sigma_{33}^{a},\ \sigma_{23}^{a}$$

$$\downarrow$$

$$\begin{cases} \varepsilon_{11}^{a} = -\frac{\nu}{E}\left(\sigma_{22}^{a} + \sigma_{33}^{a}\right) - \frac{1}{2}\frac{f\left(\sigma_{eq}^{a}\right)}{\sigma_{eq}^{a}}\left(\sigma_{22}^{a} + \sigma_{33}^{a}\right) \\ \varepsilon_{22}^{a} = \frac{1}{E}\left(\sigma_{22}^{a} - \nu\sigma_{33}^{a}\right) + \frac{f\left(\sigma_{eq}^{a}\right)}{\sigma_{eq}^{a}}\left(\sigma_{22}^{a} - \frac{1}{2}\sigma_{33}^{a}\right) \\ \varepsilon_{33}^{a} = \frac{1}{E}\left(\sigma_{33}^{a} - \nu\sigma_{22}^{a}\right) + \frac{f\left(\sigma_{eq}^{a}\right)}{\sigma_{eq}^{a}}\left(\sigma_{33}^{a} - \frac{1}{2}\sigma_{22}^{a}\right) \\ \varepsilon_{23}^{a} = \frac{(1+\nu)}{E}\sigma_{23}^{a} + \frac{3}{2}\frac{f\left(\sigma_{eq}^{a}\right)}{\sigma_{eq}^{a}}\sigma_{23}^{a} \end{cases}$$

PSEUDO-MATERIAL METHOD FOR MULTIAXIAL NON-PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- An efficient approach was proposed by **Köttgen-Barkey-Socie**, called '*Pseudo-Material*', or '*Structural Yield Surface Approach*' approach.

- It assumes that the linear elastic 'pseudo-stress' (or 'pseudo-strain') and the Elastic-Plastic notch tip strains (or stresses) can be related in all directions by a single scalar constitutive model $[\tilde{\sigma}] \times [\varepsilon]$ or $[\sigma] \times [\tilde{\varepsilon}]$, where the symbol $\sim$ denotes the *pseudo-material characteristic*.

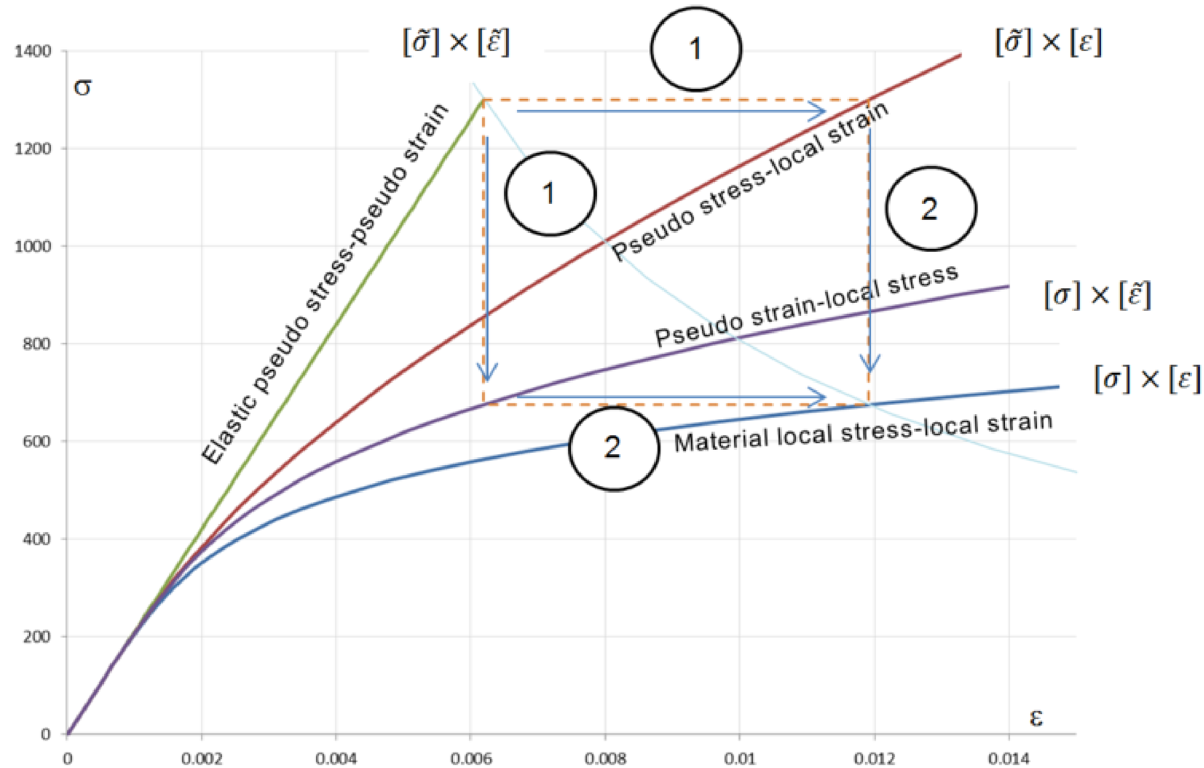
PSEUDO-MATERIAL METHOD FOR MULTIAXIAL NON-PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- The pseudo-material curve is used to simulate the Elastic-Plastic notch tip $[\sigma] \times [\varepsilon]$ behavior under multiaxial non-proportional conditions, using a *two steps calculation procedure*:
  - First step: the linear elastic pseudo-stress (or strain) loading history, obtained from FEM linear elastic analysis, is used to calculate the actual Elastic-Plastic notch tip strain (or stress) history.
  - Second step: Elastic-Plastic notch tip stress (or strain) is calculated from the Elastic-Plastic notch tip strain (or stress) history calculated at the first step.

PSEUDO-MATERIAL METHOD FOR MULTIAXIAL NON-PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

The path (1) represents the Pseudo-Stress approach, whereas the path (2) represents the Pseudo-Strain approach.

Both deliver the same final Elastic-Plastic stress-strain condition.



$$\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{K'}\right)^{\frac{1}{n'}} \qquad \tilde{\sigma} = \sqrt{\sigma^2 + E \cdot \bar{\alpha}_U \cdot \sigma \cdot \left(\frac{\sigma}{K'}\right)^{\frac{1}{n'}}} \qquad \tilde{\varepsilon} = \sqrt{\left(\frac{\sigma}{E}\right)^2 + \bar{\alpha}_U \cdot \left(\frac{\sigma}{E}\right) \cdot \left(\frac{\sigma}{K'}\right)^{\frac{1}{n'}}}$$

PSEUDO-MATERIAL METHOD FOR MULTIAXIAL NON-PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Step 1.a: discretization of the material stress-strain curve.
- Step 1.b: definition of the pseudo-stress curve discrete points:

$$\tilde{\sigma}_i = \sqrt{{\sigma_i}^2 + E \cdot \bar{\alpha}_U \cdot \sigma_i \cdot \left(\frac{\sigma_i}{K'}\right)^{\frac{1}{n'}}} \qquad \varepsilon_i = \frac{\sigma_i}{E} + \left(\frac{\sigma_i}{K'}\right)^{\frac{1}{n'}} \qquad \bar{\alpha}_U = \frac{\alpha_U + n' \cdot (2 - \alpha_U)}{1 + n'}$$

The parameter α_U spans between 1 and 2:

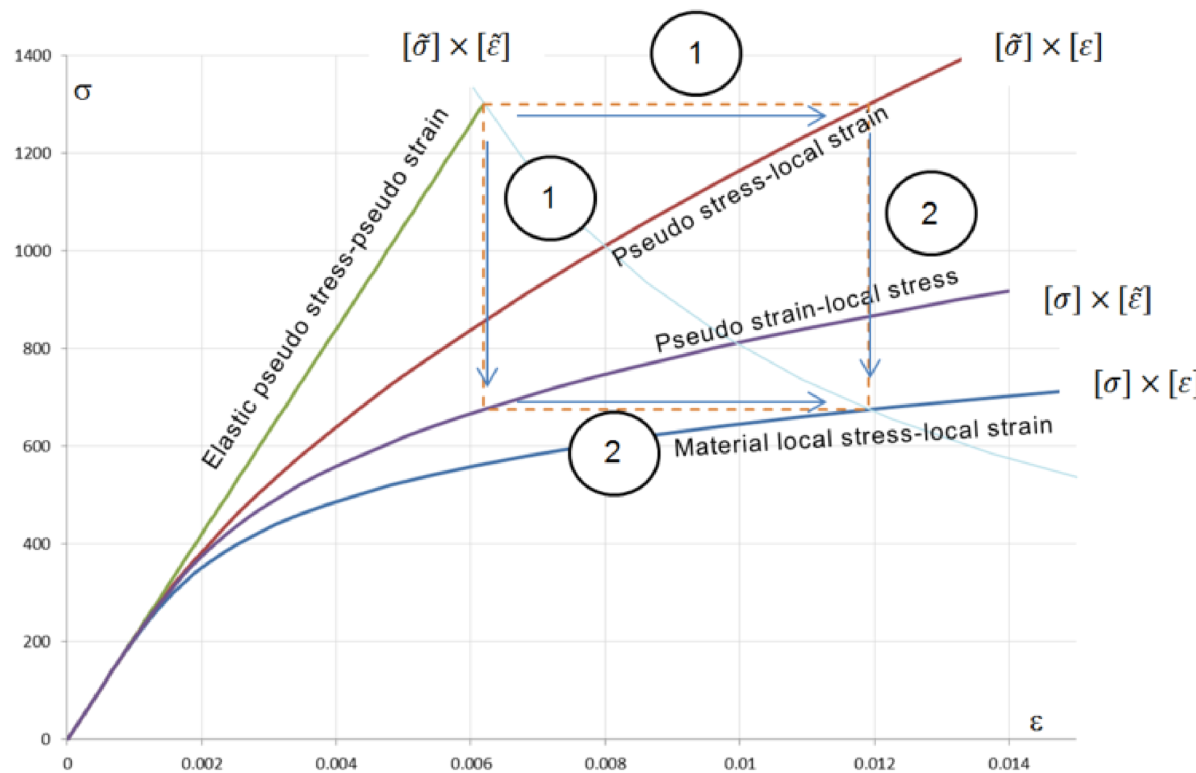
$\alpha_U = 1 \;\rightarrow\; \bar{\alpha}_U = 1 \;\rightarrow\;$ Neuber rule

$\alpha_U = 2 \;\rightarrow\; \bar{\alpha}_U = \frac{2}{1+n'} \;\rightarrow\;$ E.S.E.D. Glinka rule

Note: the pseudo-material curve $\tilde{\sigma}_i, \varepsilon_i$ cannot be fitted by a Ramberg-Osgood equation, therefore the curve is numerically given by points

PSEUDO-MATERIAL METHOD FOR MULTIAXIAL NON-PROPORTIONAL LOADING ELASTIC-PLASTIC STRESS DERIVATION

- Step 1.c: derivation of Elastic-Plastic strains history from the Pseudo-Stress curve with a *Cyclic Plasticity* model.

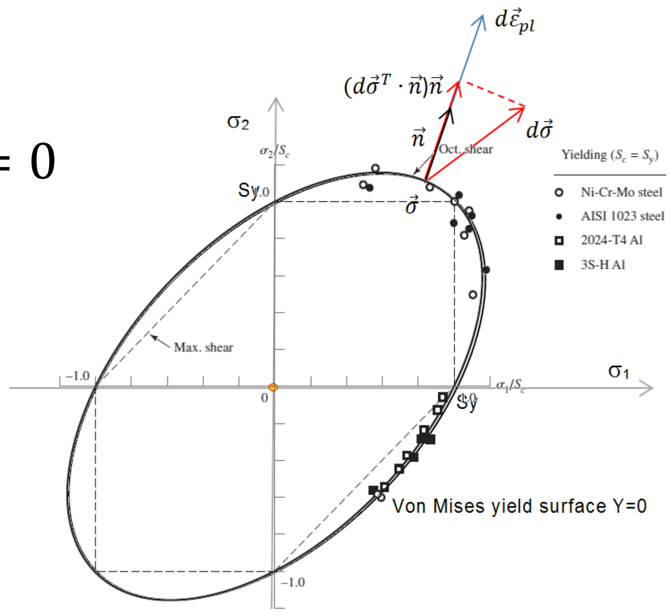


- Step 2: calculation of Elastic-Plastic stresses history from the Elastic-Plastic strains calculated in the previous step, again with a *Cyclic Plasticity* model.

CYCLIC PLASTICITY

- Cyclic Plasticity Models involve three ‘actors’:
  - Yield function

$$f = \underbrace{\sigma_1{}^2 + \sigma_2{}^2 - \sigma_1 \cdot \sigma_2}_{\sigma_{eq}} - Sy^2 = 0$$



  - Plastic Flow (“normality”) Rule

$$d\vec{\varepsilon}_{pl} = \frac{1}{C}(d\vec{\sigma}^T \cdot \vec{n}) \cdot \vec{n}$$

C = Plastic modulus

  - Hardening Rule
    - Many models have been developed, for example the *Mróz-Garud* Multi-Surface (hardening) Model

CYCLIC PLASTICITY

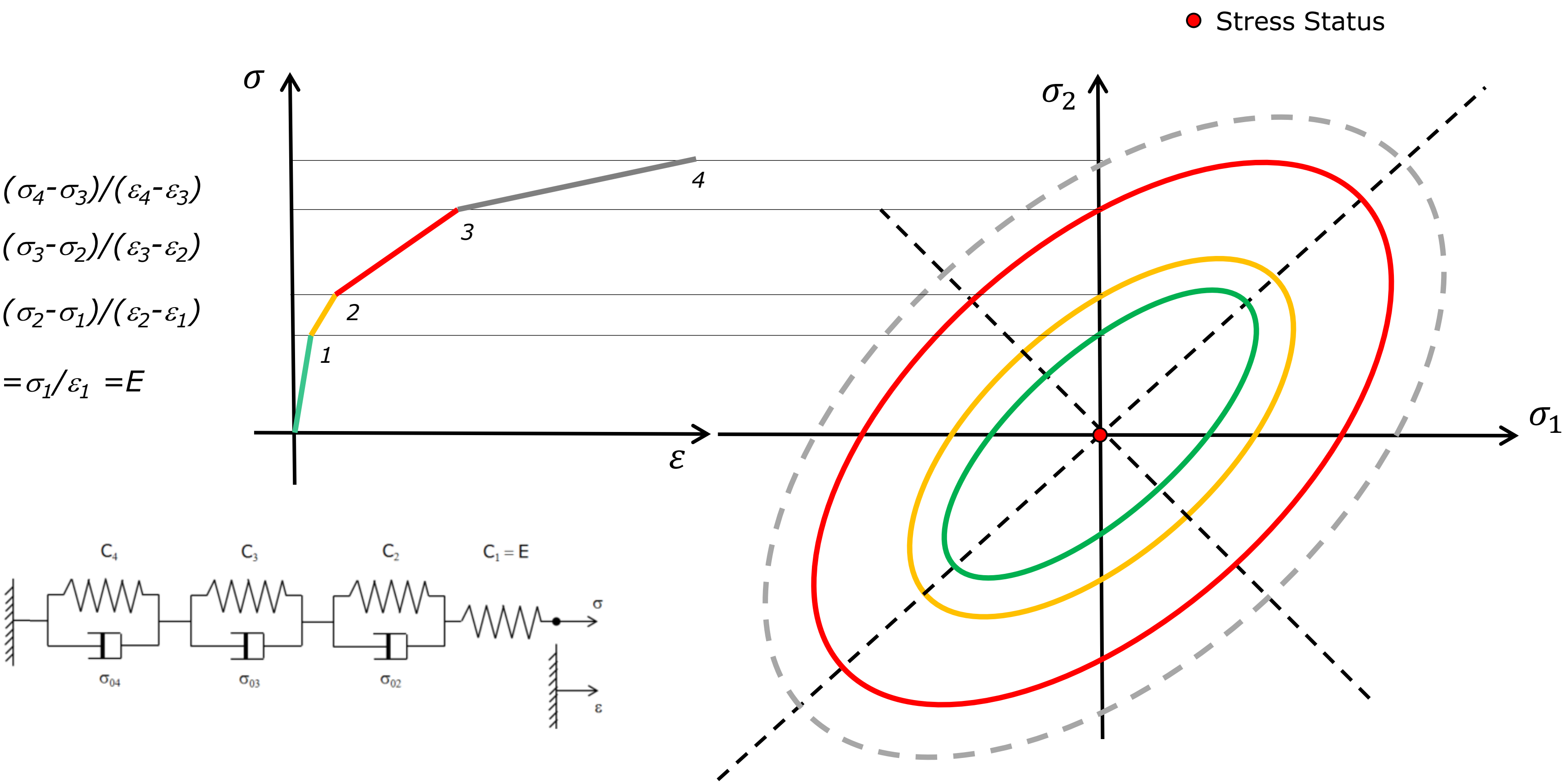
- Instant 0 $(\vec{\sigma} = \vec{0})$

$C_3=(\sigma_4-\sigma_3)/(\varepsilon_4-\varepsilon_3)$

$C_2=(\sigma_3-\sigma_2)/(\varepsilon_3-\varepsilon_2)$

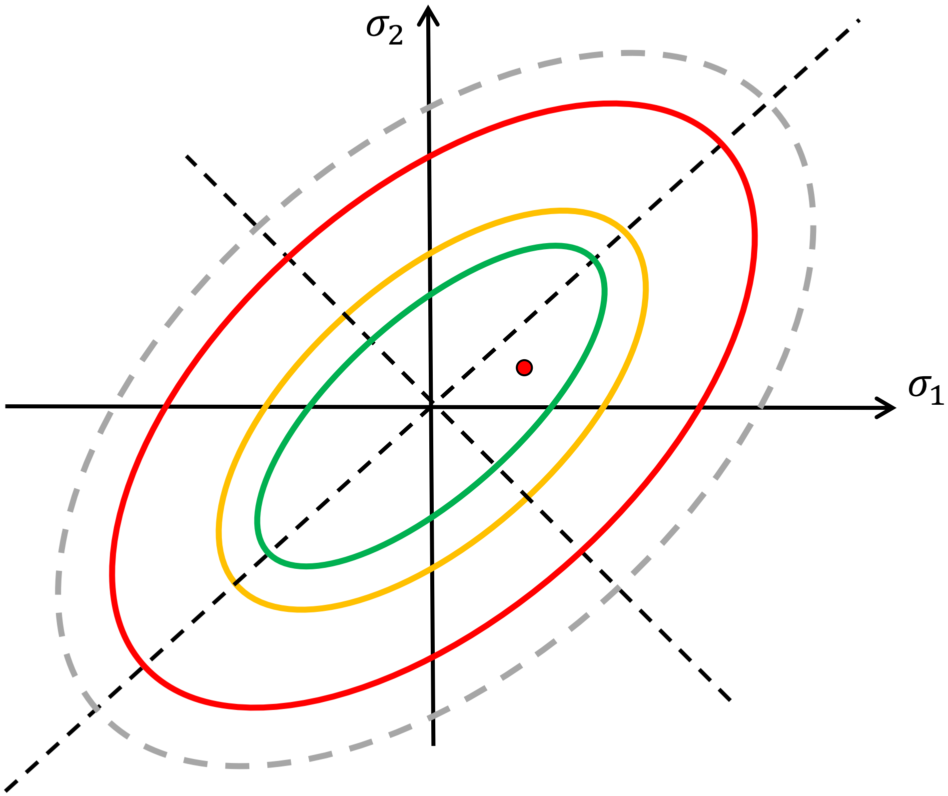
$C_1=(\sigma_2-\sigma_1)/(\varepsilon_2-\varepsilon_1)$

$C_0 = \sigma_1/\varepsilon_1 = E$



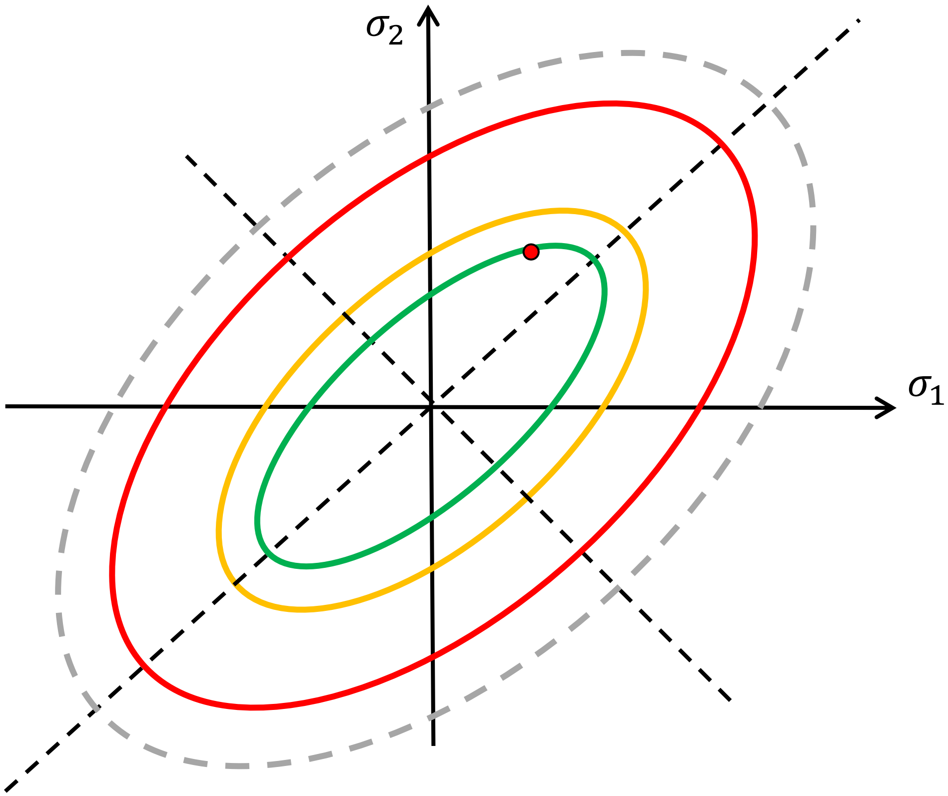
CYCLIC PLASTICITY

- Instant 0
- Instant 1



CYCLIC PLASTICITY

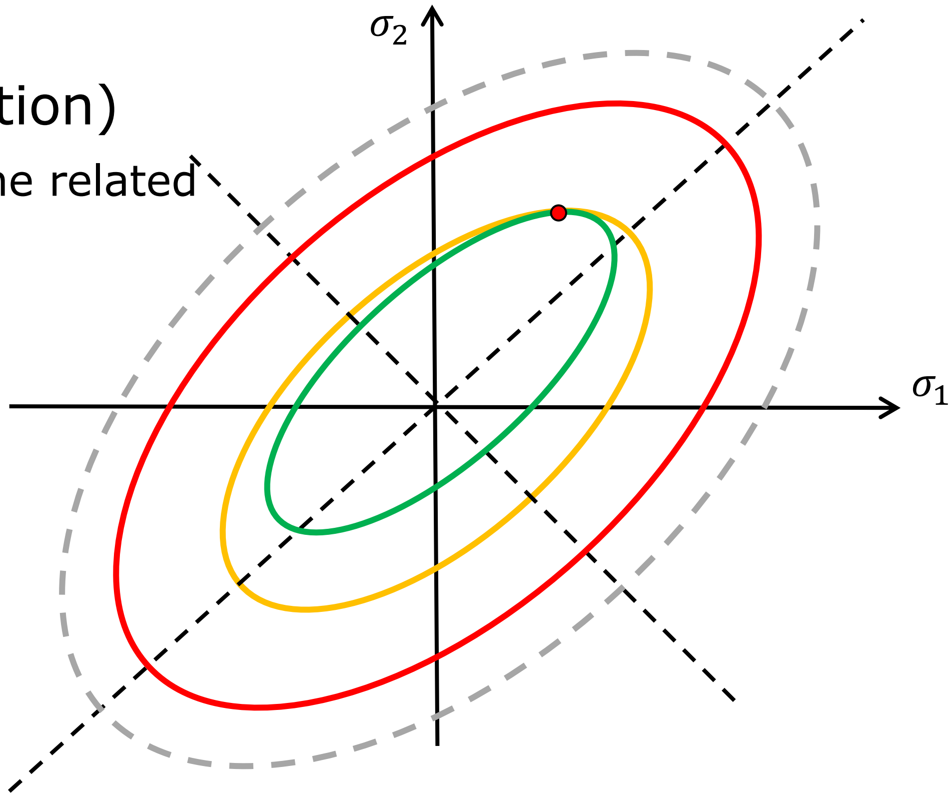
- Instant 0
- Instant 1
- Instant 2 (yielding)



CYCLIC PLASTICITY

- Instant 0
- Instant 1
- Instant 2
- Instant 3 (yielding surf. translation)

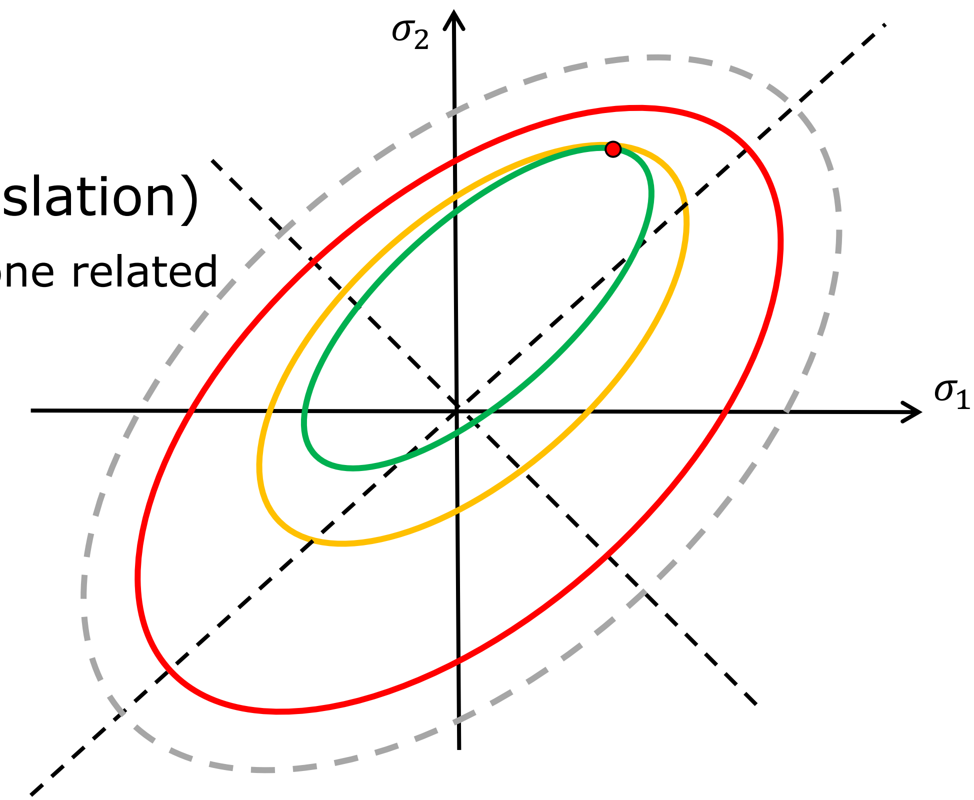
The plastic modulus C is the one related to the first surface, C1



CYCLIC PLASTICITY

- Instant 0
- Instant 1
- Instant 2
- Instant 3
- Instant 4 (hardening surf. translation)

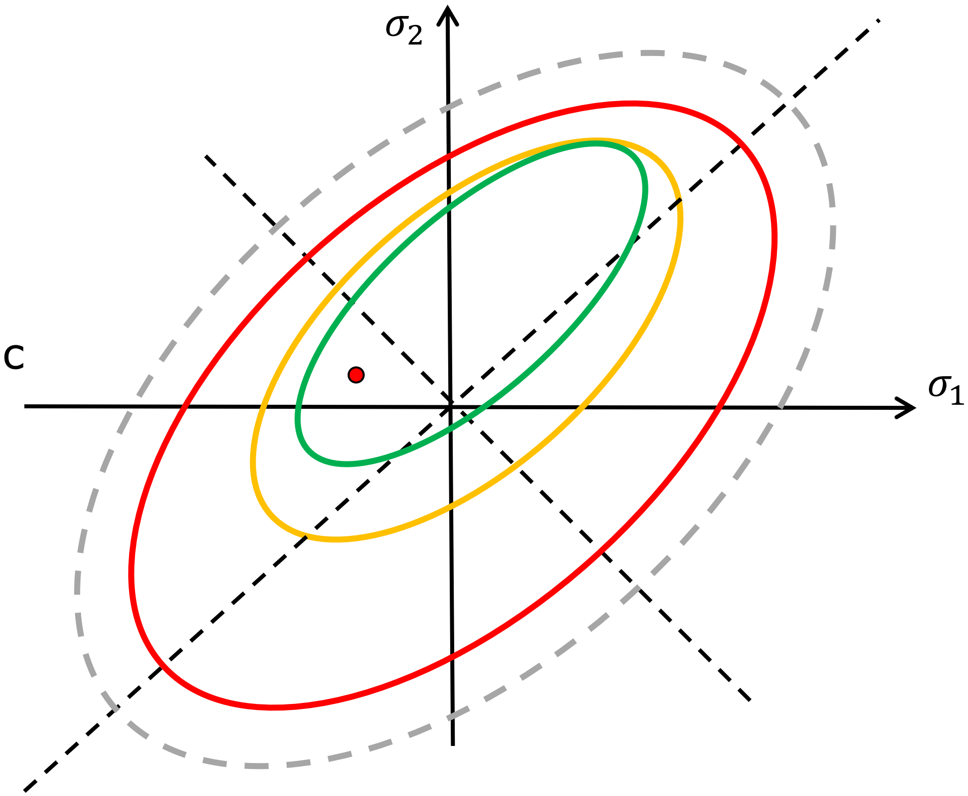
The plastic modulus C is the one related to the second surface, C2



CYCLIC PLASTICITY

- Instant 0
- Instant 1
- Instant 2
- Instant 3
- Instant 4
- Instant 5

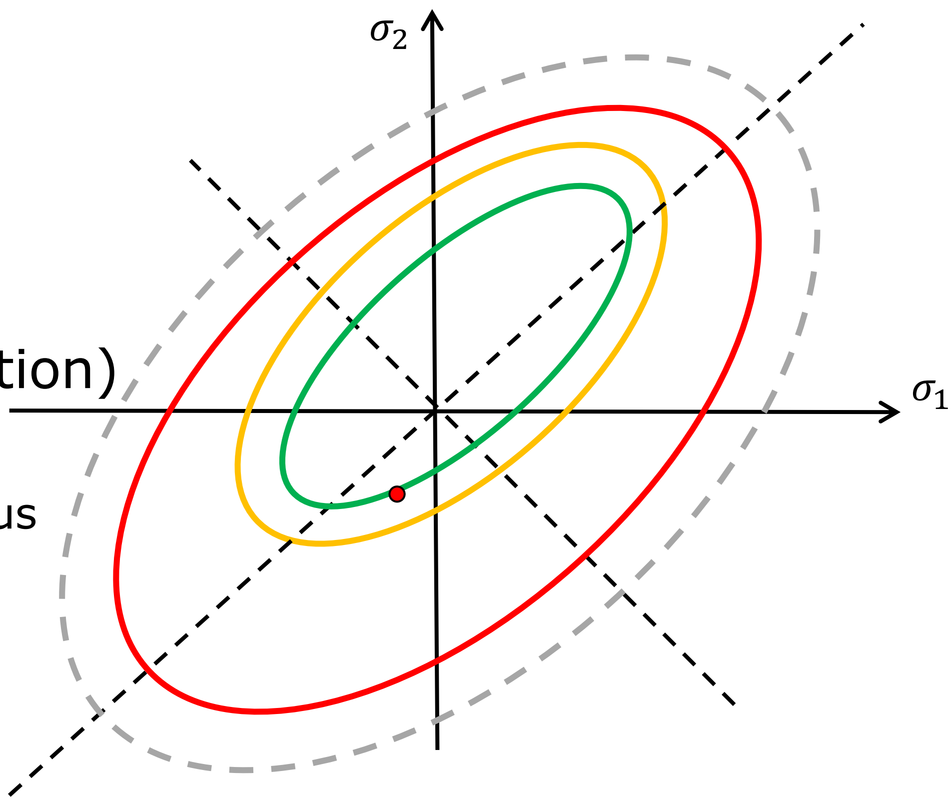
This is a ‘reversal’ in the elastic region, i.e. C=E



CYCLIC PLASTICITY

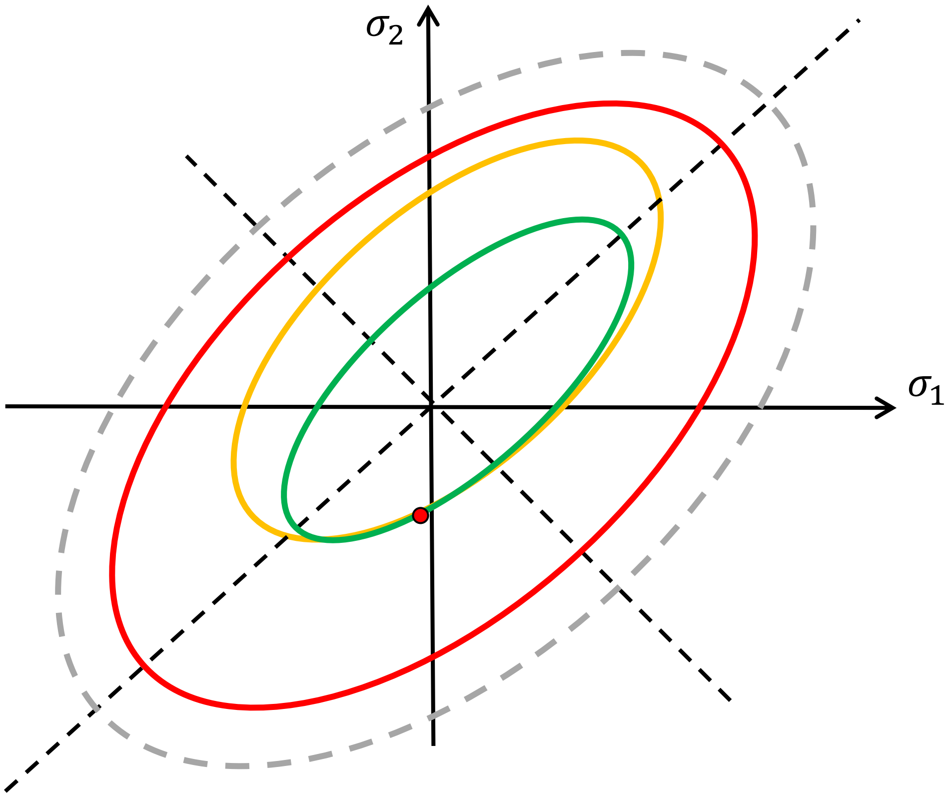
- Instant 0
- Instant 1
- Instant 2
- Instant 3
- Instant 4
- Instant 5
- Instant 6 (yielding surf. translation)

Here again the first surface is touched, i.e. the plastic modulus C is the one related to the first surface, C1



CYCLIC PLASTICITY

- Instant 0
- Instant 1
- Instant 2
- Instant 3
- Instant 4
- Instant 5
- Instant 6
- Instant 7
- …

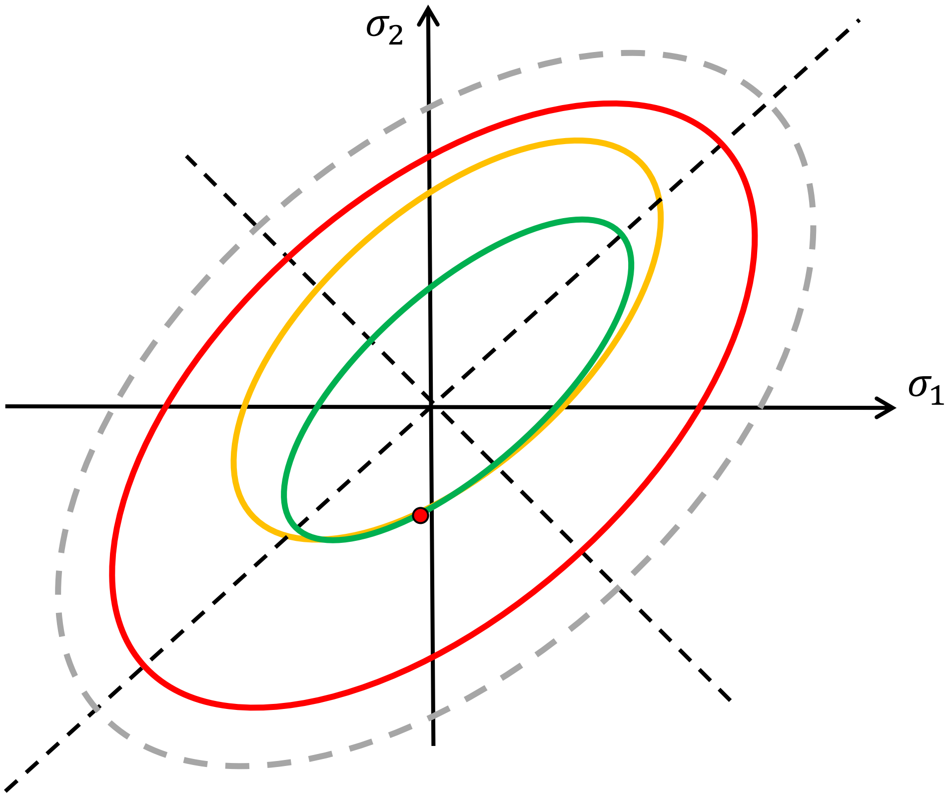


COMPUTATIONALLY DEMANDING

(INCREMENTAL APPROACH)

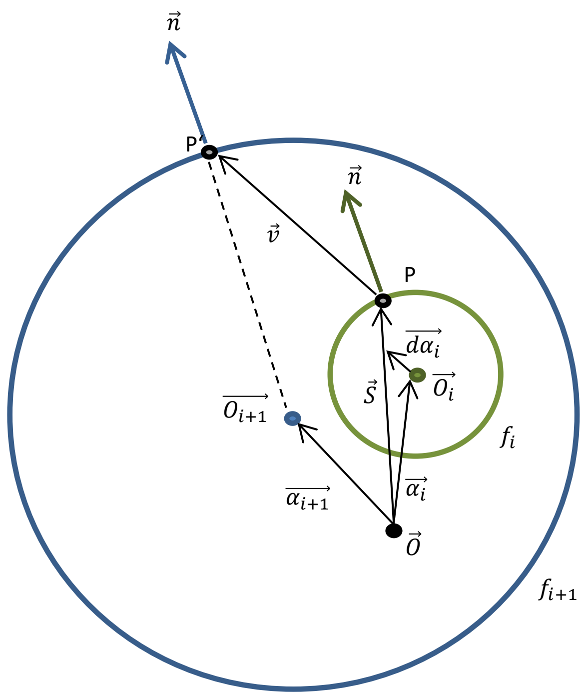
CYCLIC PLASTICITY

- It is clear from this example that the stress condition at a certain instant t depends on the time history of the previous events (which determine the disposition of the hardening surfaces)…
- …in other words the instantaneous stress condition due to a given Load depends on the previous loads time history
- This is the Memory Effect

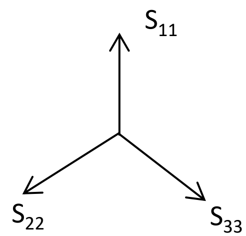


CYCLIC PLASTICITY

- Mróz hardening rule



Voigt-Mandel deviatoric space



$$\overrightarrow{d\alpha_i} = \frac{\overrightarrow{dS} \cdot \vec{n}}{\vec{v} \cdot \vec{n}} \vec{v}$$

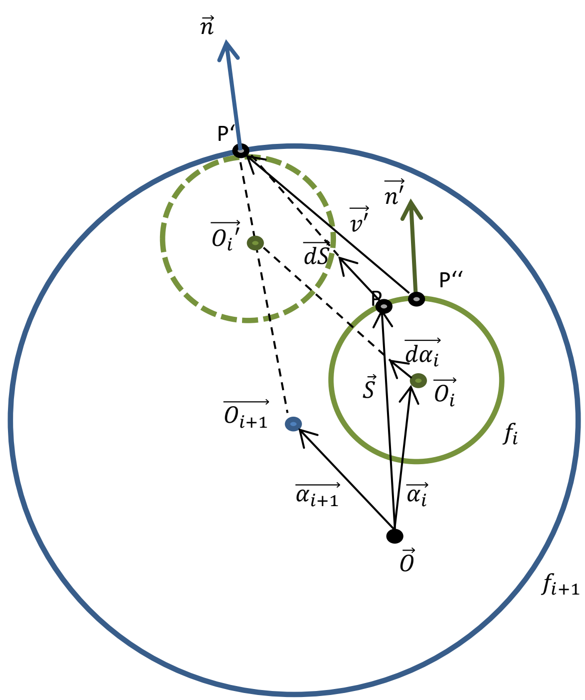
$$\vec{v} = \sqrt{\frac{2}{3}}(R_{i+1} - R_i)\vec{n} + \vec{\alpha}_{i+1} - \vec{\alpha}_i$$

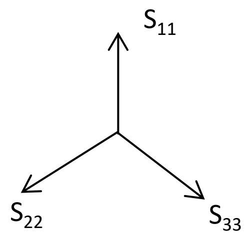
Where $\vec{\alpha}_i$ and R_i represent the center and the radius of the *i-th* surface respectively. The term $\vec{\alpha}_i$ is the '**backstress**' (center) of the *i-th* surface at the current stress state.

An important rule of the multi-surface kinematic hardening model is that, while translating, the surfaces cannot cross each other.

CYCLIC PLASTICITY

- Mróz-Garud hardening rule





Voigt-Mandel deviatoric space

$$\overrightarrow{d\alpha_i} = \frac{\overrightarrow{dS}\cdot\vec{n}}{\vec{v}'\cdot\vec{n}}\overrightarrow{v'}$$

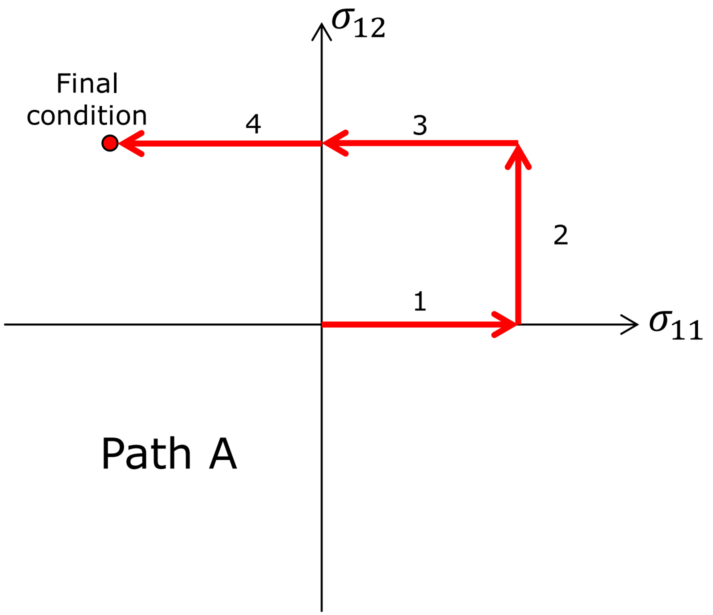
$$\overrightarrow{v'} = \sqrt{\frac{2}{3}}(R_{i+1} - R_i)\overrightarrow{n'} + \vec{\alpha}_{i+1} - \vec{\alpha}_i$$

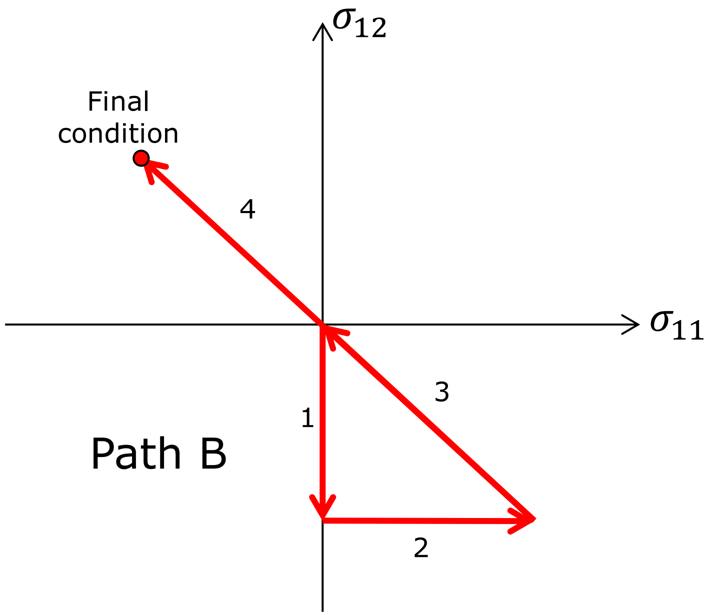
Garud, in examining hardening rules, found that when the stress increments are discretized (finite, not infinitesimal), the translation of the yield surface creates an inconsistency as the hardening surfaces may intersect each other while translating.

In order to avoid this inconsistency, Garud modified the model, where the translation is based on another vector, $\overrightarrow{v'}$, calculated such to have tangency of the surfaces at the contact point.

CYCLIC PLASTICITY

- Mróz-Garud Model Calculation Example
  - Assume two different Load paths:

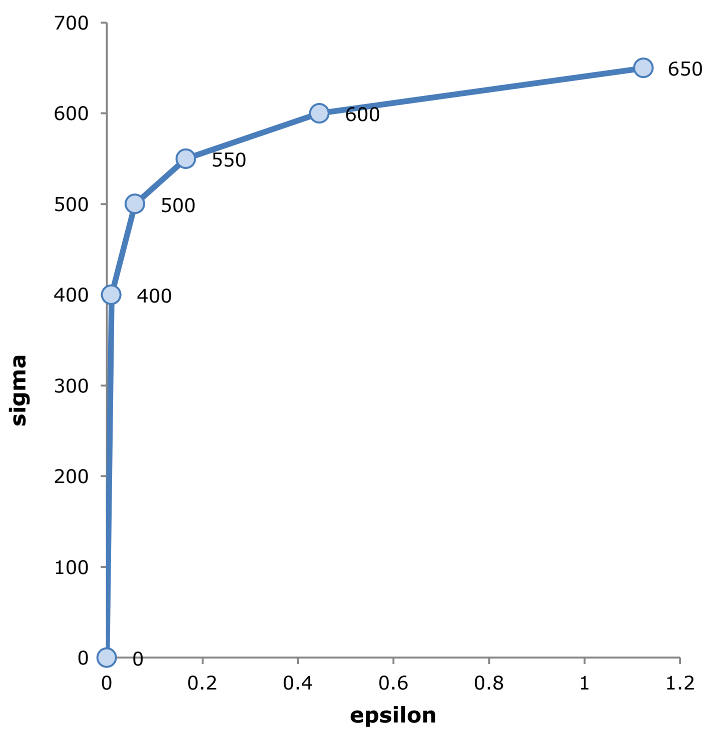


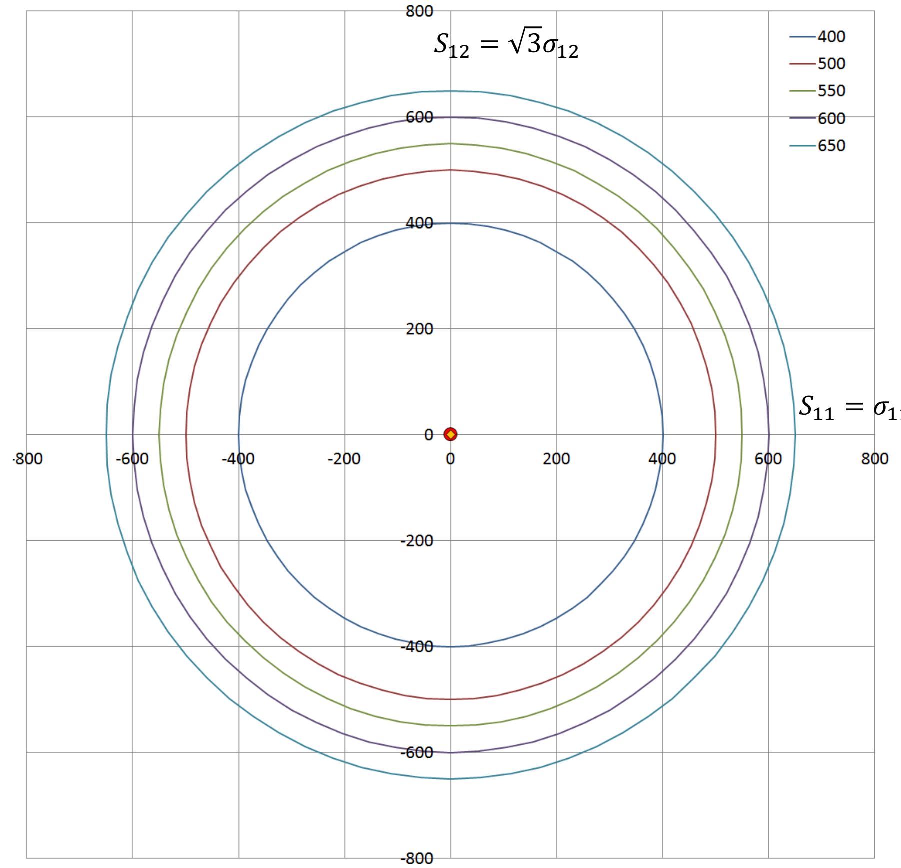


  - Different paths, same final elastic condition

CYCLIC PLASTICITY

- Mróz-Garud Model Calculation Example
  - Assume the following material curve (discretized in 5 surfaces only for simplicity, however commercial codes use 20-40 surfaces)





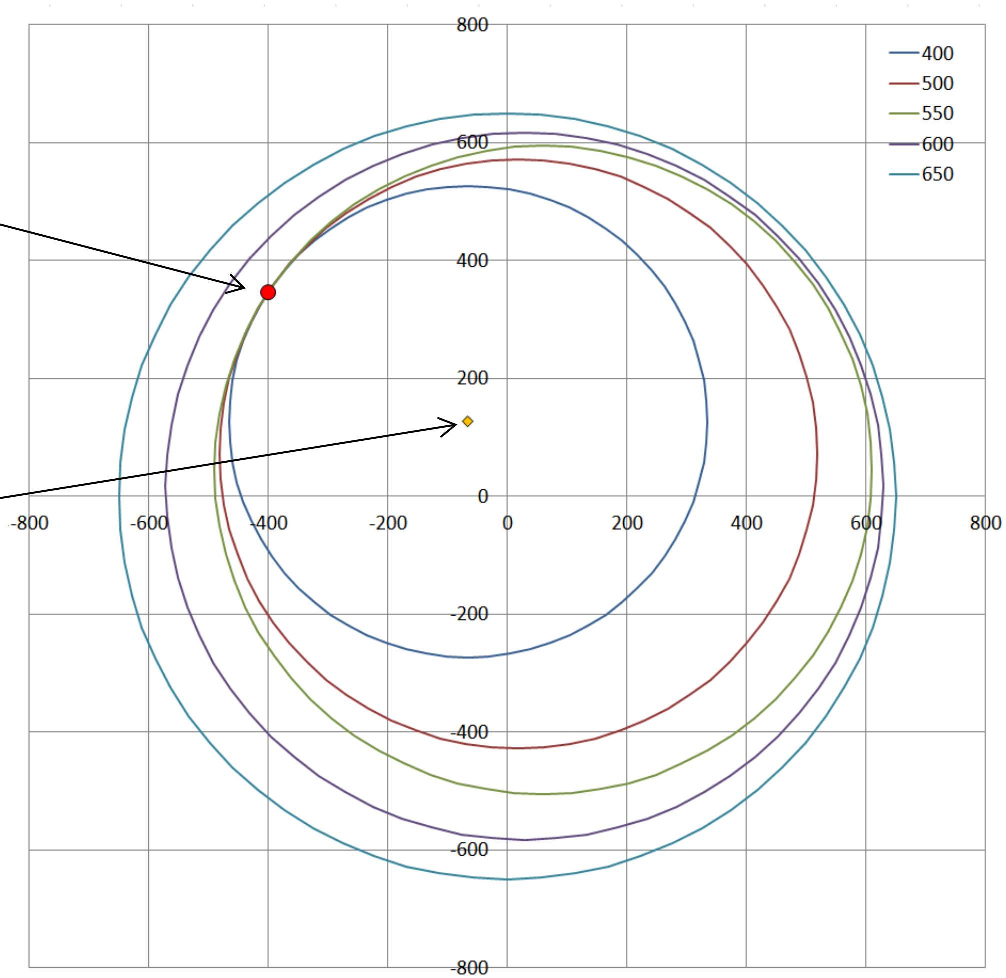
CYCLIC PLASTICITY

- Mróz-Garud Model Calculation Example – Path A

$$S_{11} = -400\ MPa$$
$$S_{12} = 346.4\ MPa$$

$$S_{11} = -65.6\ MPa$$
$$S_{12} = 127\ MPa$$

Eff. stress $\vec{S} - \vec{O}$
$$S_{11} = -334.4\ MPa$$
$$S_{12} = 219.4\ MPa$$



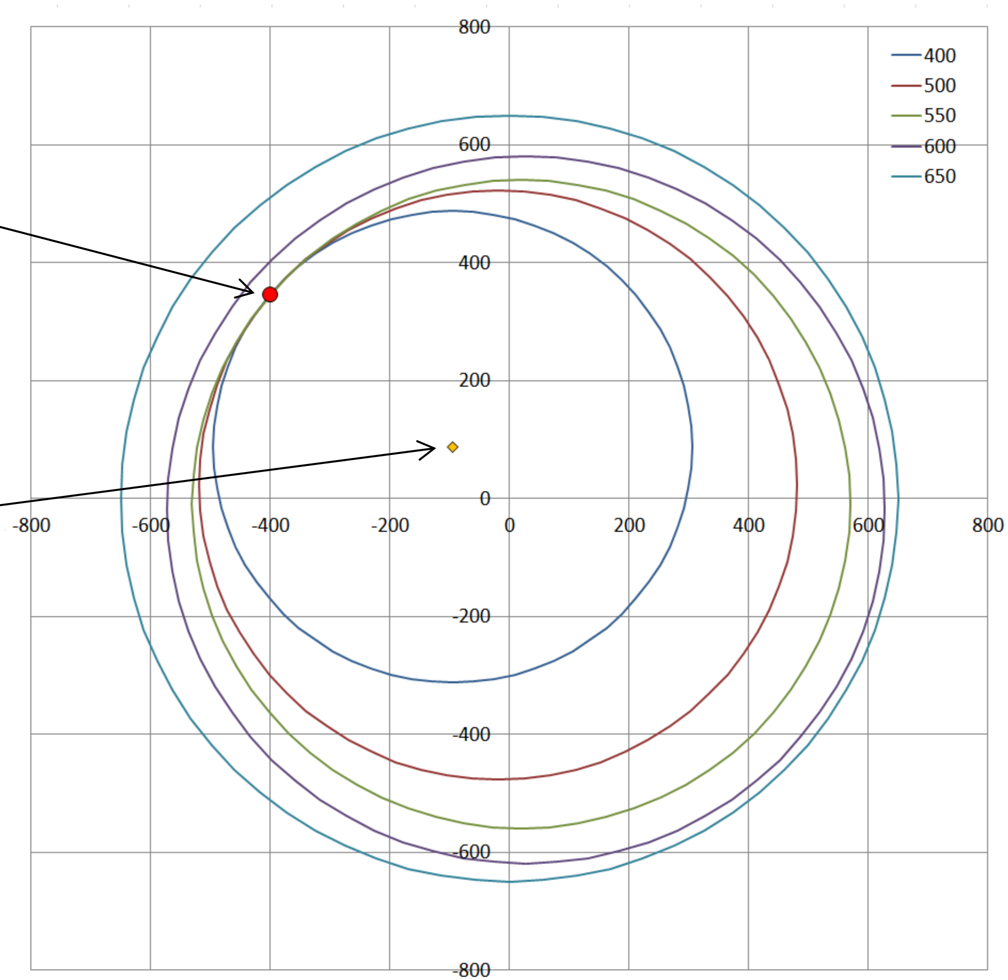
CYCLIC PLASTICITY

- Mróz-Garud Model Calculation Example – Path B

$S_{11} = -400\ MPa$
$S_{12} = 346.4\ MPa$

$S_{11} = -94.9\ MPa$
$S_{12} = 87.7\ MPa$

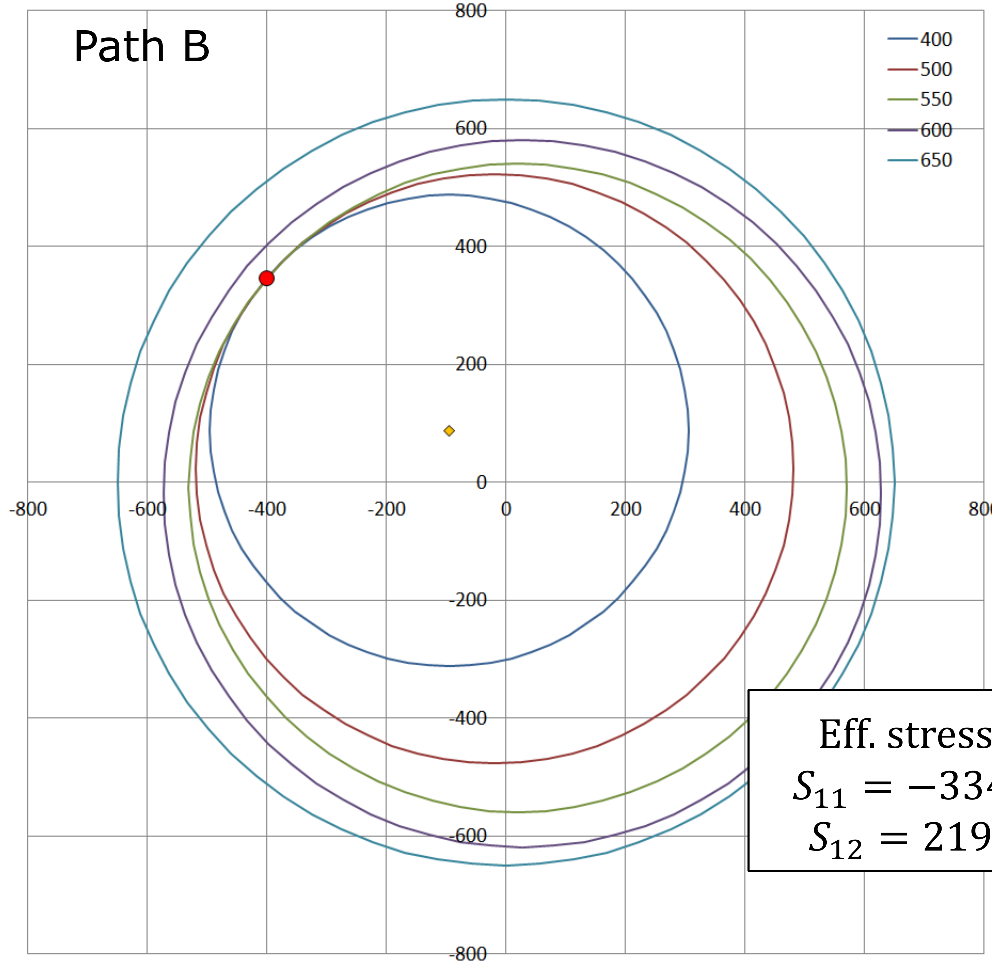
Eff. stress $\vec{S} - \vec{O}$
$S_{11} = -305.1\ MPa$
$S_{12} = 258.7\ MPa$



CYCLIC PLASTICITY

- Mróz-Garud Model Calculation Example – Path A vs Path B

  - Notice that, though the final condition is the same, the hardening surfaces have a different layout at the end
  - When calculating the Elastic-Plastic stress/strain condition the solution will be different between Path A and Path B
  - This is the Memory Effect



Eff. stress $\vec{S} - \vec{O}$

$S_{11} = -334.4\ MPa$

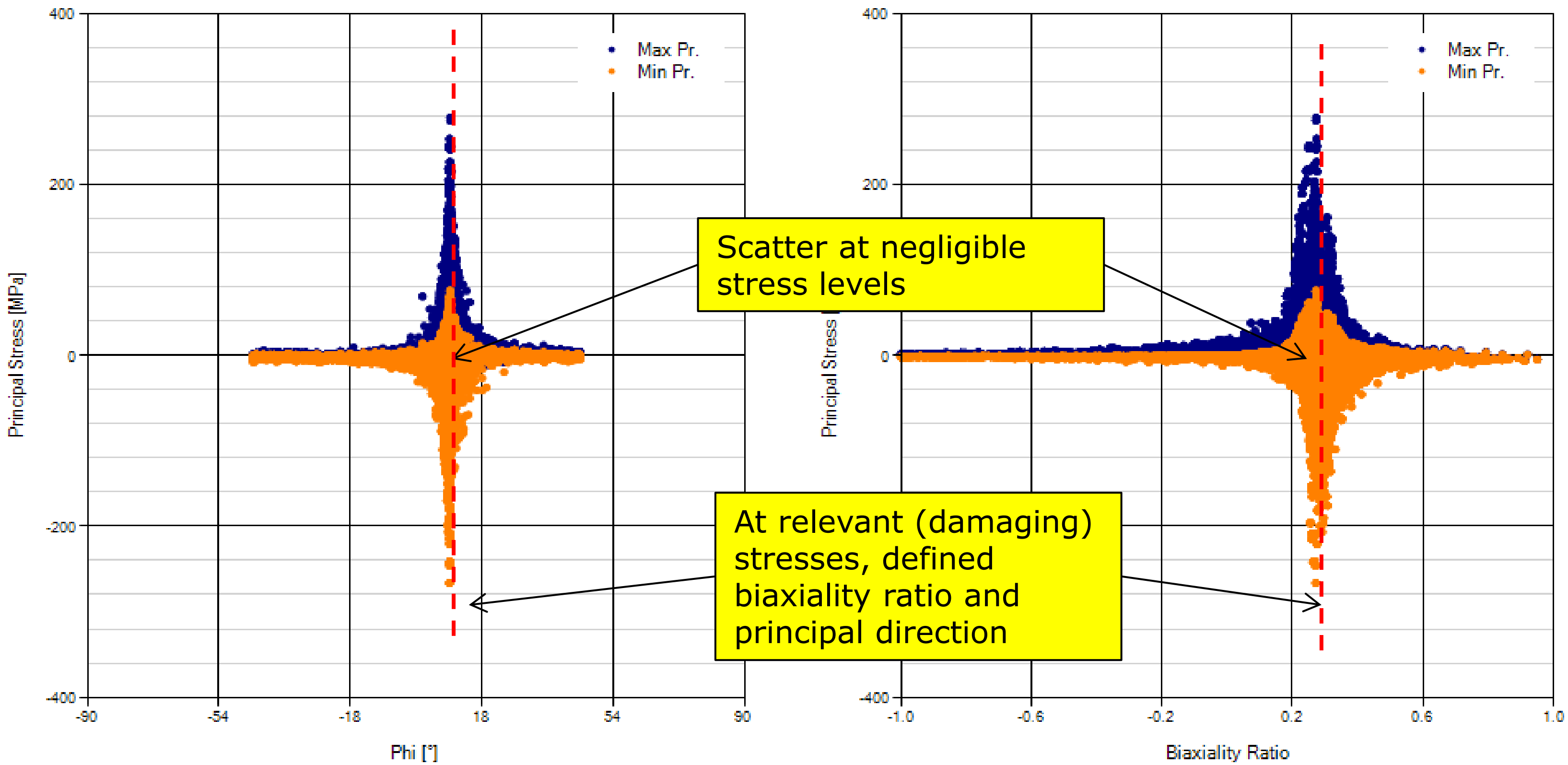
$S_{12} = 219.4\ MPa$

CYCLIC PLASTICITY

- Mróz-Garud Model Calculation Example
  - At each increment, solving the Mróz-Garud Model implies determining:
    - Which hardening surface is ‘involved’, i.e. what is the Plastic Modulus C
    - Where is the Stress Status Point
    - Where is the Backstress
  - All the above make the Plastic Flow be calculated and the Elastic-Plastic stress tensor be derived
  - In case of the Pseudo-Material approach, the model is solved twice (at step 1 and step 2)

'PROPORTIONAL REDUCTION' SIMPLIFIED METHOD

- If a multiaxial assessment is done at a real structure notch for real load time histories, we find the following



‘PROPORTIONAL REDUCTION’ SIMPLIFIED METHOD

- While building the elemental stress time history the weight average biaxiality ratio and principal direction are calculated (weight based in a power of stress or strain)

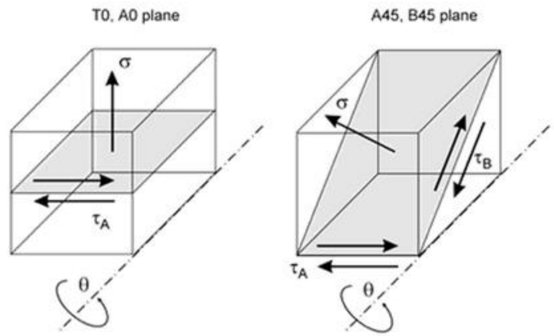
- If the variance is close to 1 the we can reduce the problem by extracting the stress at the average max principal direction and assigning the average biaxiality ratio

- Then the problem can be solved with Dowling or Hoffman-Seeger (way faster) with negligible loss of precision (engineering approach)

CRITICAL PLANE APPROACH

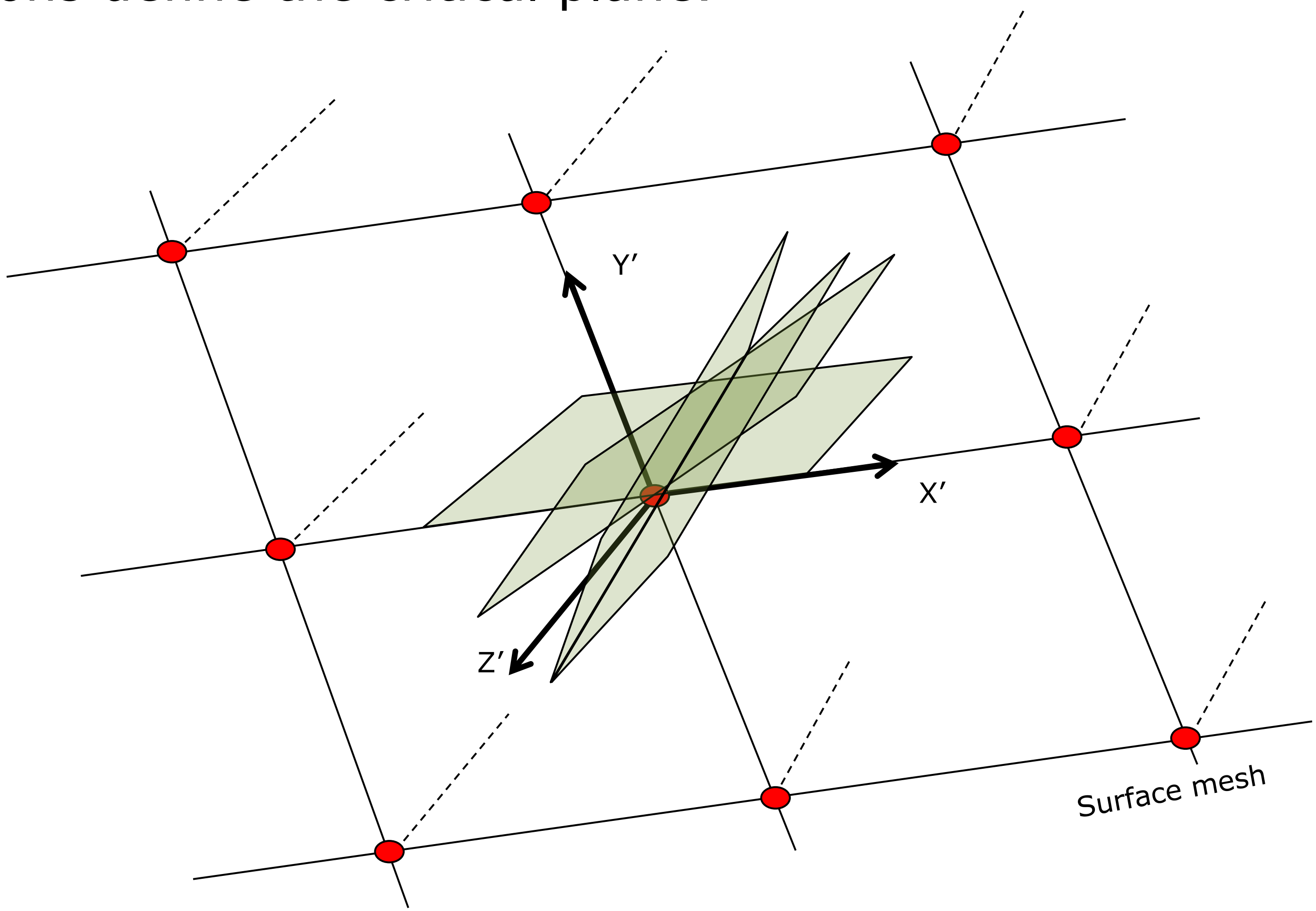
- Critical plane methods are very popular. They are based on analysis performed at many 'candidate' critical planes.

- Amongst all 'candidate' planes, the critical one is the plane which maximizes the defined fatigue parameter, e.g., in LCF:
  - Smith-Watson-Topper: $\sigma_{N,max} \cdot \frac{\Delta\varepsilon_N}{2}$
  - Brown-Miller: $\frac{\Delta\gamma_{max}}{2} + S\Delta\varepsilon_N$
  - Fatemi-Socie: $\frac{\Delta\gamma_{max}}{2}\left(1 + S\frac{\sigma_{N,max}}{S_y}\right)$



CRITICAL PLANE APPROACH

- Working on the surface of a mechanical component, two rotations define the critical plane.



CRITICAL PLANE APPROACH

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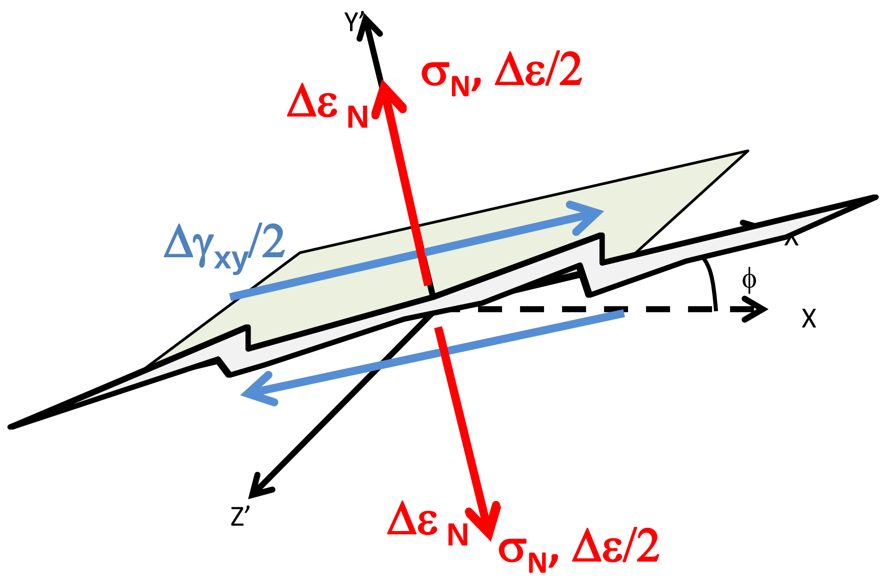
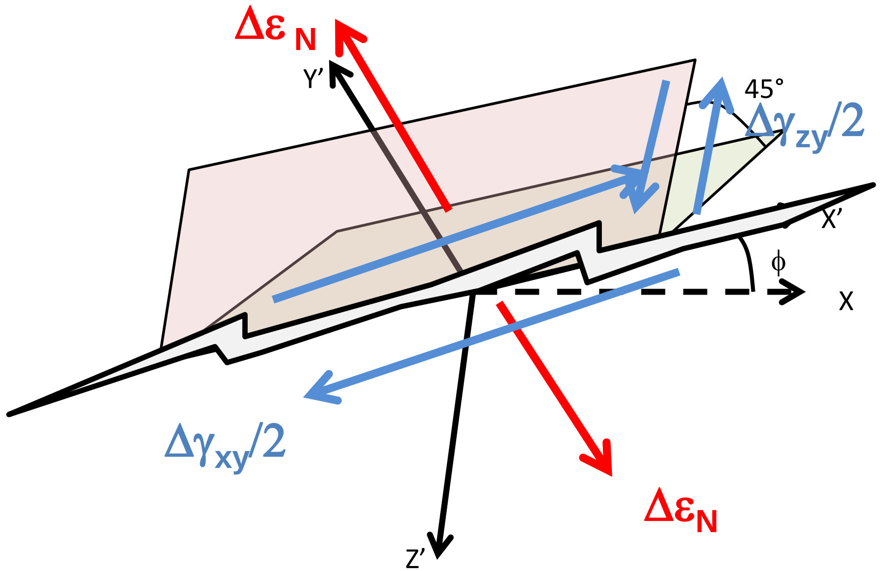


SWT – Plane T0

BM/FS – Plane A0

CRITICAL PLANE APPROACH

- Working on the surface of a mechanical component, two rotations define the critical plane.

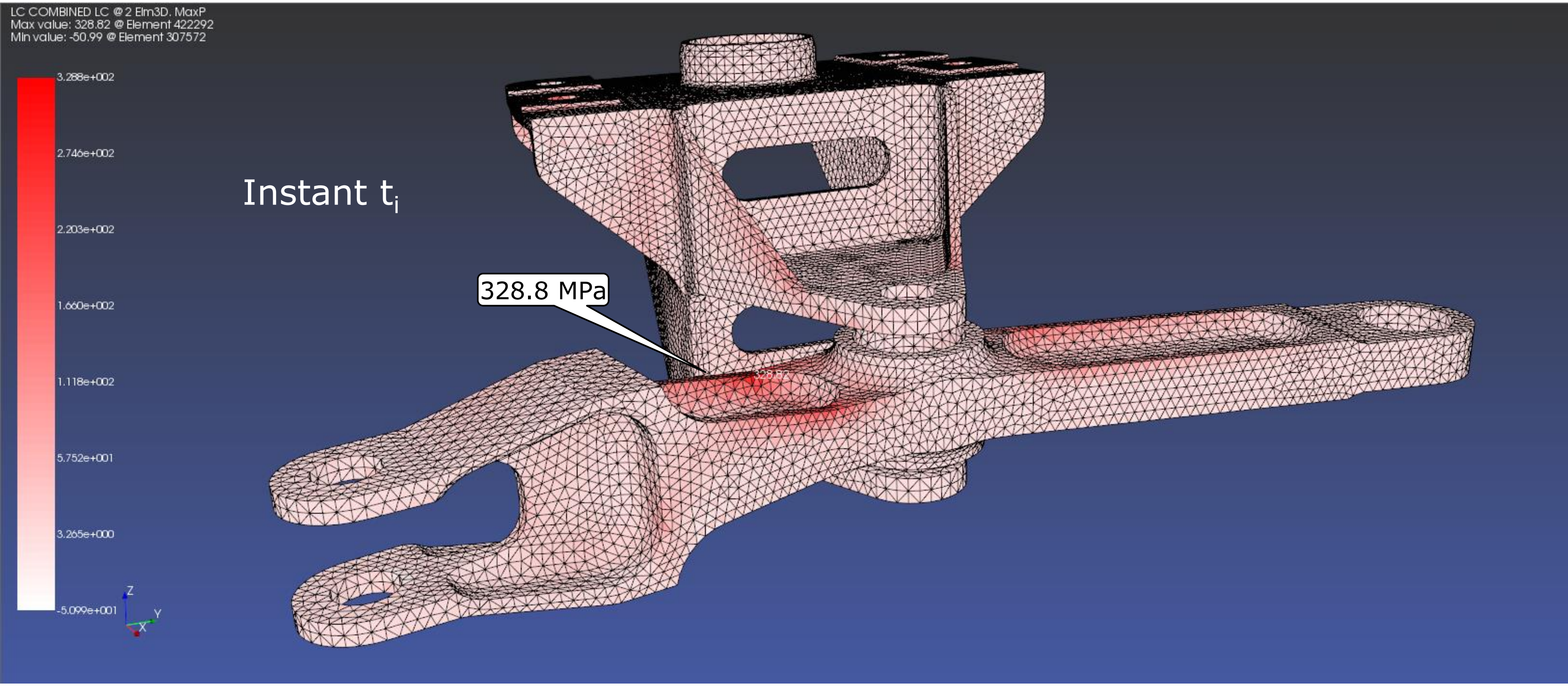


BM/FS– Plane A45

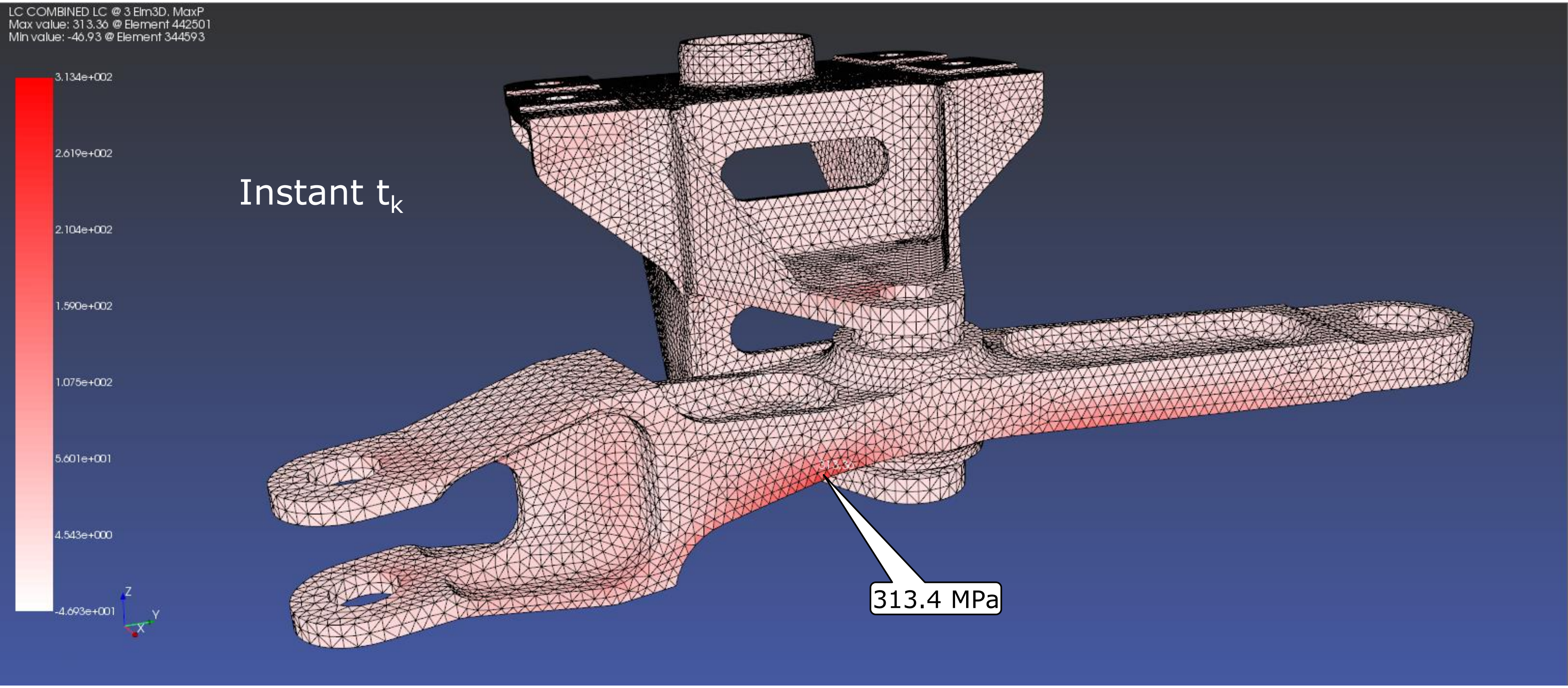
BM/FS– Plane B45

- The second rotation is usually done at 45° only

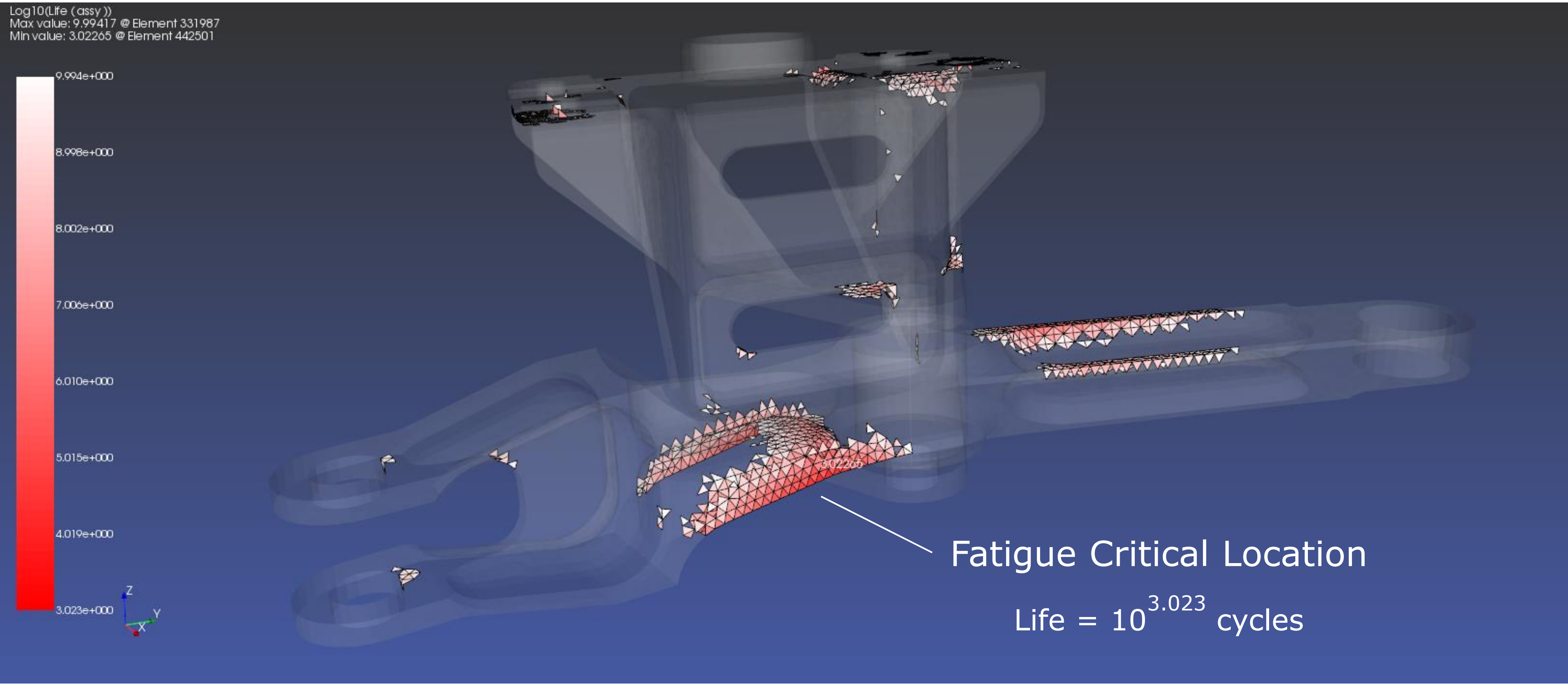
EXAMPLE WITH THE SOFTWARE LIFING



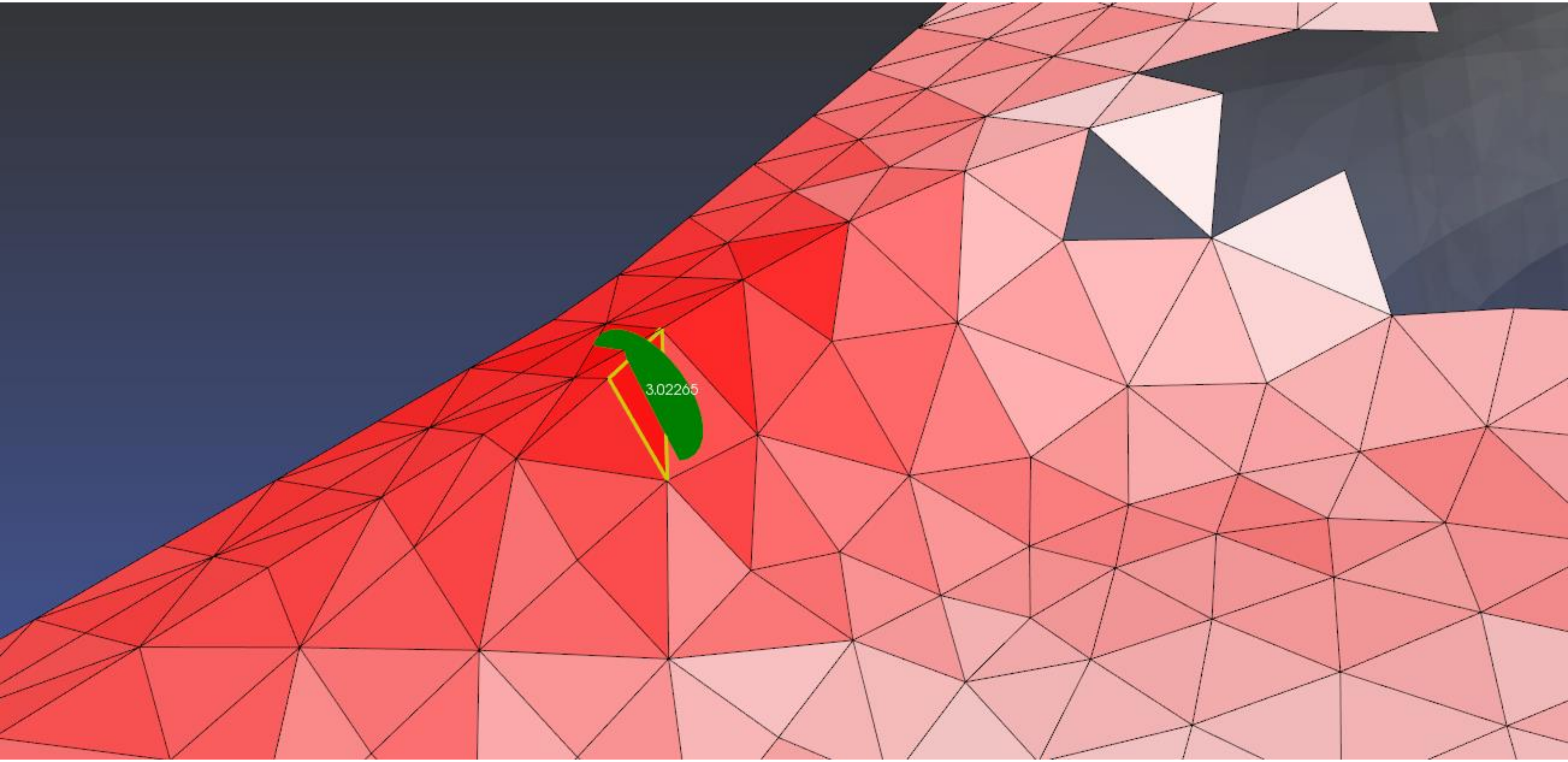
EXAMPLE WITH THE SOFTWARE LIFING



EXAMPLE WITH THE SOFTWARE LIFING



EXAMPLE WITH THE SOFTWARE LIFING



DANG VAN HCF METHOD

- The Dang Van approach is a multiscale fatigue crack initiation approach for HCF based on the use of mesoscopic stresses, i.e. stresses at the grain scale, where slip bands are identified(*).

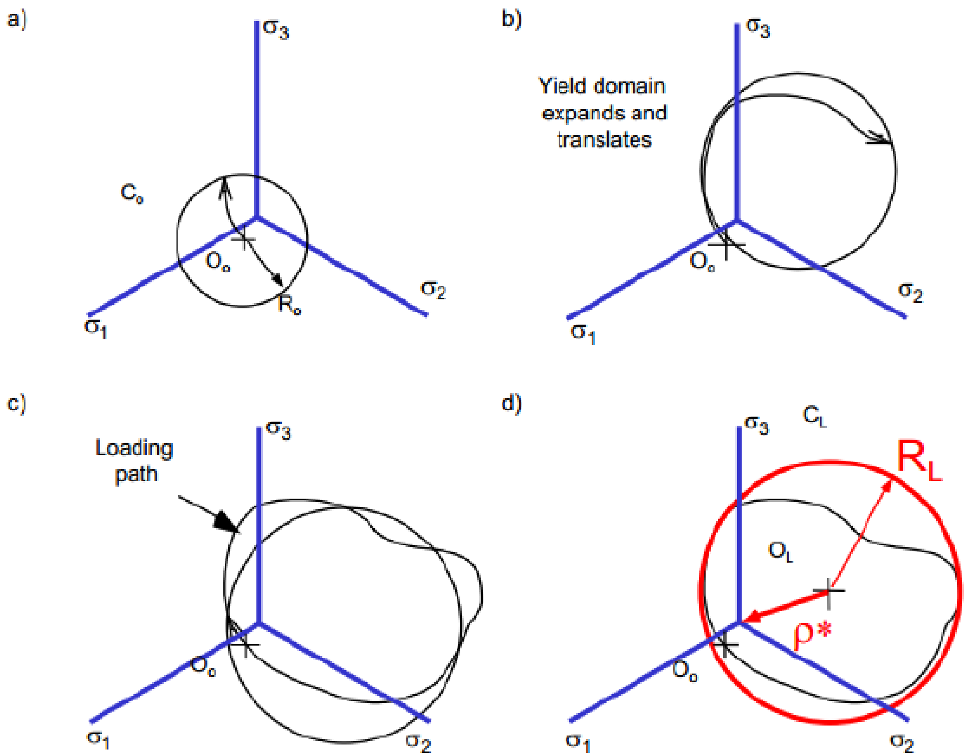
- (*) This grain scale is intermediate, between the engineering macroscopic scale (i.e. crack propagation scale or, in FEM, the elements scale) and the microscopic scale (i.e. where dislocations occur).

DANG VAN HCF METHOD

- Dang Van postulated the following:
  - For an infinite lifetime, near the fatigue limit, crack nucleation in slip bands may occur in the most unfavourably oriented grains, which are subjected to plastic deformation even if the macroscopic stress is elastic. Residual stresses in these plastically deformed grains will be induced due to the restraining effect of the adjacent grains.
  - An elastic shakedown (i.e. stabilization of elastic response) in a macroscopic state occurs before the fatigue limit and both the mesoscopic and macroscopic plastic strains and residual stresses are stabilized.

DANG VAN HCF METHOD

- The initial elastic domain of the critical volume of material is illustrated by the circle Co, with center at Oo and radius Ro (at beginning equal to 0)
- As loading progresses the material undergoes combined kinematic and isotropic hardening, as the center of the yield surface translates and the radius of the surface increases.
- After several repetitions of the load path, a stable domain CL with center OL and radius RL will evolve.
- The stable path is characterized by the smallest circle, CL that completely encloses the load path.

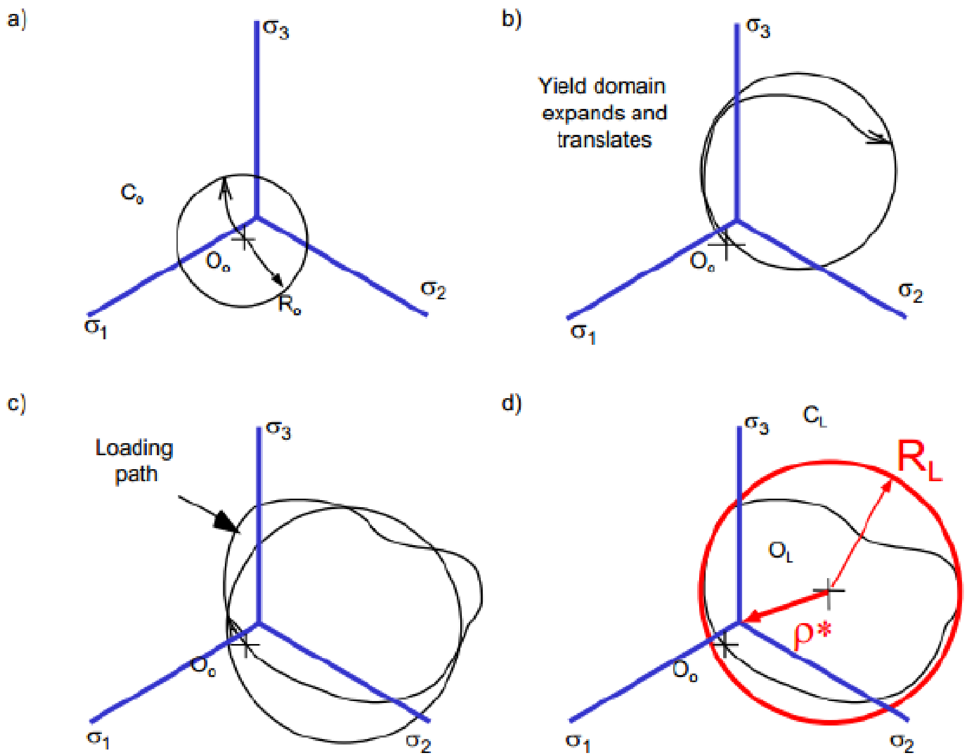


DANG VAN HCF METHOD

- Dang Van postulates that the stabilized residual stress tensor corresponds to the center of such a circle, representing the smallest Von Mises yield surface, that completely encloses the path described by the deviatoric stress tensor
- It is given by the min-max function

$$[\rho]^* = Min_\alpha\{Max_t[\sigma_{VM}(t)]\}$$

- Which is calculated iteratively



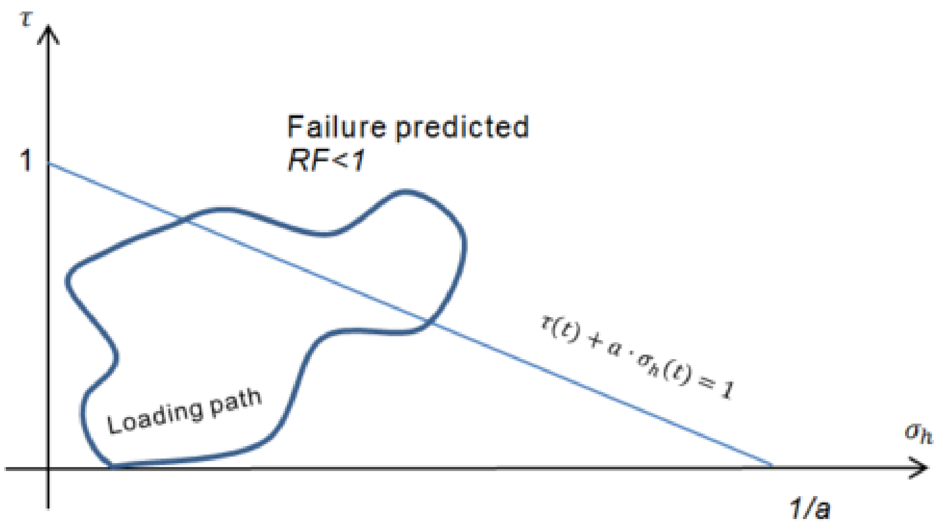
DANG VAN HCF METHOD

- According to the Dang Van criterion, fatigue damage does not occur if

$$Max_t[|\tau_{meso}(t) + a \cdot \sigma_{meso-h}(t)|] \leq b$$

- where *a* is the hydrostatic stress sensitivity and *b* is the shear fatigue limit.
- At each instant t, the following reserve factor is calculated

$$RF = \frac{b}{|\tau_{meso}(t) + a \cdot \sigma_{meso-h}(t)|}$$



THE NEW CHALLENGES

- Why the need of removing conservativism? ‘Fatigue analyses have always been done with conventional conservative approaches’...

- ...true, but the industry is going in the direction of ‘super-optimized’ structures. Converntional techniques are INSUFFICIENT! State-of-the-art analysis methods and tools are required

REFERENCES

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- Metal Fatigue in Engineering – Stephens-Fatemi-Stephens
- Multiaxial Fatigue, Socie-Marquis
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THANK YOU

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